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WORKING PAPER

# 26-17 Debt Sustainability and the Range of Multiple Equilibria

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## ABSTRACT

We characterize the size of the range of multiple equilibria in sovereign debt markets. We show that the nature of the equilibrium depends crucially on the degree of uncertainty, which we formalize first as uncertainty about primary balances and later as uncertainty about the fiscal feedback rule. For low uncertainty, the range can be large, but it becomes smaller as uncertainty becomes larger. If uncertainty is large enough, the multiplicity disappears. The interest rate is then unique and becomes a continuous function of the debt level. Interpreting the source of an increase in the interest rate in actual situations is difficult and complicates the use of intervention by central banks.

**JEL Code:** H6

**Keywords:** debt, sustainability, equilibria, interest rates, uncertainty

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# 1 Introduction

In the face of high public debt levels and sometimes high primary deficits, economists sound the alarm. Under current policies, they argue that, in a number of countries, debt does not appear to be sustainable. When asked, however, how and when this will lead to a debt crisis, they become more timid and are reluctant to give a date.

The reason behind this timidity is, in part, the presence of multiple equilibria. The interest rate investors require to hold the debt depends on their perceived probability of default. And debt dynamics, and thus the probability of default, depend in turn on the interest rate investors require. At very low debt levels, debt is clearly sustainable. At very high debt levels, debt is clearly unsustainable. But in between, there is a range of debt levels where there are multiple equilibria. Here, we focus on the two extreme cases. There is one in which investors are rationally optimistic and require a low interest rate, and debt is likely to be sustainable. We call it the good equilibrium. And one in which investors are rationally pessimistic and require a higher rate, and debt is likely not sustainable. We call it the bad equilibrium.

Within this range, both equilibria are rational expectation equilibria, and what determines whether the economy stays at the good equilibrium or shifts to the bad one is largely beyond the ability of economists to assess.

The purpose of this paper is to clarify the complex relation of the interest rate to the level of debt, and the size and determinants of the range of multiple equilibria. We choose a simple analytical framework, too simple to be seriously calibrated, but simple enough to identify the mechanisms at work.

Our main result is that the range of multiplicity is decreasing with the uncertainty about fundamentals. The reason is that lower uncertainty makes the actual probability of default more sensitive to investors' beliefs, which is precisely what sustains multiple equilibria. As uncertainty rises, this sensitivity falls, the feedback weakens, and the range shrinks.

Related literature Our contribution is not to show that sovereign debt markets can have multiple equilibria. Calvo (1988) and Cole and Kehoe (2000) established that self-fulfilling debt crises can arise when the government lacks commitment and the debt level lies inside a crisis zone, so that reducing debt enough to exit that zone may be optimal. The same logic underlies second-generation models of currency crises (Obstfeld, 1994). Subsequent

work has shown how rich these crises can be. Lorenzoni and Werning (2019) show that a self-fulfilling crisis need not take the form of a sudden run but can be rather slow-moving. Aguiar et al. (2022) show that such crises need not end in default at all, and may instead show up as debt issued at steeply depressed prices. Ayres et al. (2018) study how expectations select among equilibria, and Bocola and Dovis (2019) quantify how much of euro-area spreads reflected rollover rather than fundamental risk.

Our contribution is to show that aggregate uncertainty is the key determinant of the *size* of the range over which multiple equilibria exist, the analogue of the crisis zone in that literature. The mechanism is the familiar one: beliefs about default shape the interest rate, the interest rate shapes debt dynamics, and debt dynamics validate or refute the initial beliefs. Our point is that the strength of that feedback depends on the degree of uncertainty about fundamentals. To isolate this channel, we deliberately model an economy without a strategic government, in which the primary surplus is exogenous and stochastic. Our main result, that uncertainty shrinks the range of multiplicity, echoes Frankel and Pauzner (2000), who show that indeterminacy in a dynamic economy with strategic complementarities is fragile to the introduction of aggregate shocks.

Our model belongs to a second strand of the literature, which sees default as unavoidable rather than strategic, and characterizes the resulting limits on sustainable debt. Whether because of political constraints or of a Laffer curve maximum on revenue, debt eventually becomes unsustainable and default ensues. Multiplicity takes the same basic form as above: over some range there are two equilibria, one in which investors ask for a low rate and debt is sustainable, and one in which they ask for a higher rate, debt is unsustainable, and default happens. This approach, largely empirical, dominates policy discussions of debt dynamics and of the evolution of spreads and ratings.

The starting point of this strand is Bohn (1995, 1998), which characterizes the condition for debt sustainability, although without discussing multiplicity. Ghosh et al. (2013) consider a government that follows a feedback rule but eventually suffers from fatigue; they note that the pricing equation can have several solutions, but impose assumptions that select a unique one, and focus on the resulting debt limit. Bi (2012) models the debt ceiling as the present value of maximum primary surpluses at the peak of a state-dependent Laffer curve, Bi and Traum (2012) estimate the implied fiscal-limit distribution for Greece and Italy, and Bi et al. (2013) study fiscal consolidations whose size and timing are uncertain. Collard et al. (2015) characterize the maximum sustainable debt under uncertainty about fundamentals. Instead of looking at the maximum sustainable debt, we focus on characterizing the range

of multiplicity and how it depends on aggregate uncertainty.

The global games literature has studied other sources of uncertainty and their effect on multiplicity. Morris and Shin (1998) depart from common knowledge and model investors who observe noisy private signals about fundamentals and do not know other investors' beliefs. When the noise is small enough, multiplicity disappears and a unique equilibrium is selected. Goldstein and Pauzner (2005) apply the same logic to bank runs. He and Xiong (2012) obtain a unique equilibrium in a dynamic debt run through a different route, staggered debt maturities together with a publicly observed, time-varying fundamental, rather than through private signals. In our model, information is common, and uncertainty is about aggregate fundamentals. A decrease in uncertainty then widens the range of multiplicity rather than eliminating it.

A growing literature studies whether monetary policy can rule out self-fulfilling sovereign debt crises (Corsetti and Dedola, 2016; Bacchetta et al., 2018; Aguiar et al., 2015; Bianchi and Mondragon, 2022). This work takes multiplicity as given and studies what interventions are welfare-improving. Our analysis is complementary. By characterizing when multiplicity exists, it speaks to whether an intervention is warranted in the first place. Furthermore, we show that a fiscal feedback rule strong enough to stabilize debt even at the rate required by pessimistic investors eliminates multiplicity. A rule that stabilizes debt only at the safe rate does not. Uncertainty about the strength of the rule itself, instead, shrinks and eventually eliminates the multiplicity that arises from beliefs about the adjustment program, provided investors are optimistic once the program is over.

Outline Section 2 introduces the basic model and considers the case of certainty. Section 3 introduces uncertainty about future primary balances. Section 4 introduces a feedback rule à la Bohn (1998), allowing primary balances to respond to debt, Section 5 studies some extensions to the baseline models. Section 6 offers some conclusions about the conduct of monetary policy in that environment.

# 2 The Case of Certainty

Assume investors are risk-neutral, and debt is one-period, rolled over over time. If there is default on the debt, the haircut is equal to $\theta < 1$.

Let the rate on debt be equal to $R$, and let the safe rate be equal to $r$. If risk-neutral

investors believe that the probability of default at the end of the period is $P^e$, then $R(P^e)$ must be such that the expected rate of return is equal to the safe rate:

$$(1 + R(P^e))(1 - P^e) + (1 + R(P^e))(1 - \theta)P^e = (1 + r),$$

so

$$1 + R(P^e) = \frac{1 + r}{1 - \theta P^e}.$$

Denote by $R$ the value of $R(P^e)$ for $P^e = 1$, so $R = (r + \theta)/(1 - \theta)$.

Let $b$ be the ratio of debt to GDP and assume zero GDP growth. Let $s$ denote the primary balance, which, in this section, we shall assume to be positive and constant. Then, debt dynamics are given by:

$$b(+1) = \frac{1 + r}{1 - \theta P^e} b - s. \tag{1}$$

Finally, assume that default takes place when the probability that debt keeps increasing forever becomes equal to one.

The law of motion (1) describes the debt that would be owed under full repayment. Default is evaluated on that path, and is a terminal event in our economy. At settlement, investors receive a fraction $1 - \theta$ of what they are owed, and we do not follow the economy after the restructuring. Furthermore, the recovery value $1 - \theta$ is not the value of a claim on a debt path that keeps exploding, which would be zero. It is the outcome of a restructuring, which we do not model and take as given.

We can now characterize the solution and the range of multiple equilibria. We focus on the two extreme cases. In the first, investors are optimistic and conjecture $P^e = 0$ at every level of debt; we call this the good equilibrium. In the second, investors are pessimistic and conjecture $P^e = 1$; we call this the bad equilibrium. When the two coincide, the equilibrium is unique, and there is no multiplicity.

**Proposition 1.** *Under certainty, if investors are optimistic, the probability of default jumps from zero to one at $b^G = s/r$. If investors are pessimistic, the probability of default jumps from one to zero at $b^B = s/R$. The range of multiplicity is thus given by $[s/R, s/r]$.*

The equilibrium is represented in Figure 1.

Note that, not surprisingly, if $\theta > 0$, then $b^G > b^B$. The range of multiple equilibria is given by the interval $\left[b^B, b^G\right]$, with size equal to $\left(\frac{1}{r} - \frac{1}{R}\right) s = \frac{\theta(1+r)}{(r+\theta)r} s$. For debt less than $b^B$, the probability of default is zero, even if investors are pessimistic. For debt greater than $b^G$,

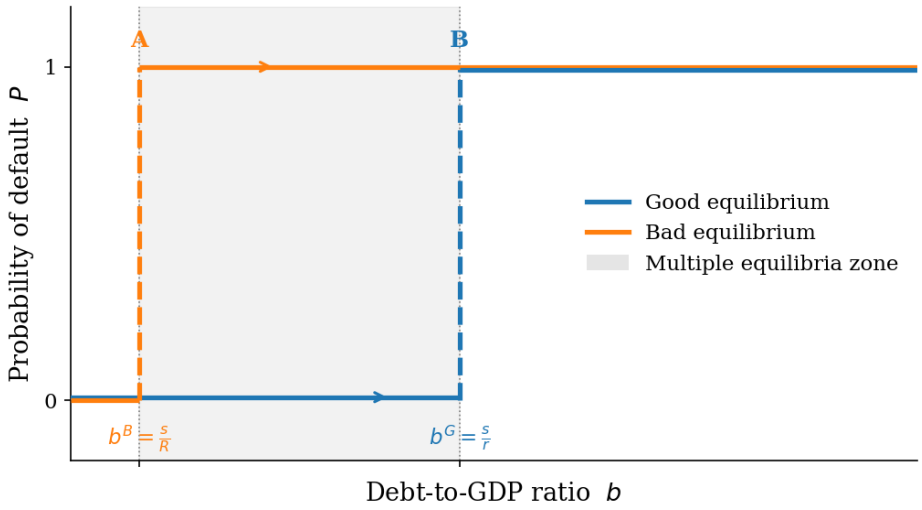



**Figure 1:** Multiple equilibria under certainty

the probability of default is one, even if investors are optimistic.

The derivation is as follows:

Suppose that investors believe $P^e = 0$. What is the highest value of debt, $b_0^G$, such that their belief is rational? That is, such that debt decreases with probability one, and therefore $P = 0$.

Derive first the value of $b$ such that debt in the next period is no higher than debt this period. Assuming $P^e = 0$ in equation (1), $b\,(+1) = b$ yields $b_0^G = \frac{s}{r}$.

Note that for values of debt greater than $b_0^G$, $b\,(+1) > b$, and by implication, $b\,(+i+1) > b\,(+i)$ for $i \geqslant 0$, so that debt explodes and thus $P = 1$. Note that for values of debt smaller than $b_0^G$, $b\,(+1) < b$, and by implication, $b\,(+i+1) < b\,(+i)$ for $i \geqslant 0$, so debt never explodes and, thus, $P = 0$. Thus, $b_0^G = \frac{s}{r}$ is the highest value of debt consistent with no default. At $b_0^G$, $P$ jumps from 0 to 1. If we define $b_1^G$ as the value of debt such that the probability of default becomes 1, $b_1^G = b_0^G = s/r$.

Suppose instead that investors believe $P^e = 1$. What is the lowest value of debt, $b_1^B$, such that their belief is rational? That is, such that debt increases with probability one, and therefore $P = 1$.

Derive the value of $b$ such that debt next period is no lower than debt this period. Assuming $P^e = 1$ in equation (1), $b\,(+1) = b$ yields $b_1^B = (1-\theta)s/(r+\theta) = s/R$.

Note that for values of debt greater than $b_1^B$, $b\,(+1) > b$, and by implication $b\,(+i+1) >$

$b(+i)$ for $i \geqslant 1$, so debt explodes, and $P = 1$. Note that for values of debt less than $b < b_1^B$, $b(+1) < b$ and, by implication, $b(+i+1) < b(+i)$ for $i \geqslant 1$, so that debt never explodes, and $P = 0$. Thus, $b_1^B = s/R$ is the lowest value consistent with default. At $b_1^B$, $P$ jumps from 1 to 0. If we define $b_0^B$ as the value of debt such that the probability of default becomes zero, $b_0^B = b_1^B = s/R$.

The model is too simplistic to do a credible calibration. But, as an exercise, think of the unit period as 5 years (roughly the average maturity of debt in advanced and emerging economies). Take the haircut rate, $\theta$, to be 0.10,[1] the interest rate to be 0.05 (1% per year) and the primary balance to be 0.1 (2% per year). Then, $b^G = 2$, $b^B = 0.6$. This suggests a large range over which multiple equilibria can occur, indeed a range covering debt ratios in most advanced economies today.

Note two characteristics of the equilibrium: A range of multiple equilibria; and discontinuous jumps of the probability of default, and by implication of the interest rate. Outcomes become richer in both dimensions when we introduce uncertainty.

# 3 Introducing Uncertainty

Assume that the primary balance is no longer constant but is equal to $s + \epsilon$, with $\epsilon$ following a uniform distribution on $[-e, +e]$, $e < s$. The choice of a bounded distribution, which implies that, for some values of debt, the probability of default may be strictly equal to zero or one, allows for an analytical characterization of the solution, making the nature and the determinants of the solution clearer. The bounded support is also part of the default criterion. With an unbounded upper tail, no finite level of debt would make the explosion of debt certain, and the default rule above would have to be replaced by another one. The other assumptions are the same as before.[2]

[1] In the dataset of sovereign debt restructurings of Ando et al. (2024), covering 57 countries over 1991–2021, the median face value reduction was 0.06 and the mean was 0.10. The face value reduction is measured as a share of the debt outstanding in the year before the restructuring, and covers private external, private domestic, and official Paris Club creditors. We thank Prachi Mishra for providing these numbers.

[2] Imposing a uniform prior is the least informative specification consistent with the surplus lying in the interval $s \pm e$, adding no structure beyond the range itself. We later discuss how results hold under non-uniformity.

Debt dynamics are thus given by:

$$b\left(+1\right)=\frac{1+r}{1-\theta P^{e}}b-\left(s+\epsilon\right). \tag{2}$$

The debt law describes the debt that would be required under full repayment. We impose default at settlement whenever this debt enters a region from which it would increase forever with probability one under the selected continuation schedule. This reduced-form rule can be interpreted as a refinancing constraint: investors do not provide the funds required for full repayment once debt enters that region.

The key to deriving the solution is the following function, $F\left(b,b^{*},P^{e}\right)$, giving the probability that, given current value $b$, and perceived probability $P^{e}$, debt exceeds some reference value $b^{*}$:

$$F\left(b,b^{*},P^{e}\right):=Prob\left(b\left(+1\right)\geqslant b^{*}\right|P^{e}\left)=clip_{[0,1]}\left\{\frac{1}{2e}\left(\frac{1+r}{1-\theta P^{e}}b-\left(s-e\right)-b^{*}\right)\right\}, \tag{3}$$

where the expression inside the brackets is the probability implied by the uniform distribution before it is bounded between zero and one. In the region where $F$ is strictly between zero and one, its derivative with respect to $P^{e}$ is:

$$F'\left(b,b^{*},P^{e}\right)\equiv\frac{\theta}{2e}\frac{1+r}{\left(1-\theta P^{e}\right)^{2}}b. \tag{4}$$

Now we are ready to define a rational expectations equilibrium for this economy.

**Definition 1.** *A **rational expectations equilibrium** is a value of debt $b^{*}$ and a function $P^{*}:b\rightarrow[0,1]$ such that*

1. *For any $b$, the perceived probability of default is equal to the actual probability of default. That is, $P\left(b\right)$ solves*

$$P^{*}\left(b\right)=F\left(b,b^{*},P^{*}\left(b\right)\right)\qquad\forall b. \tag{5}$$

2. *$b^{*}$ is the smallest value of debt such that, if $b>b^{*}$, debt increases forever with probability one under the rate schedule implied by $P^{*}$.[3] That is,*

$$b^{*}=inf_{x}\left\{Prob\left(b(+1)>b\right|P^{*}\left(b\right)\left)=1\qquad\forall b>x\right\}.$$

[3]We do not use $b\geqslant b^{*}$ because, as we shall see next, there are some equilibria in which $P^{*}\left(b^{*}\right)<1$ but $P^{*}\left(b\right)=1$ for $b>b^{*}$.

In words, an equilibrium in this economy is characterized by a debt limit $b^*$ and the fixed point $P = P^e$ such that the perceived law of motion is equal to the actual law of motion, and debt increases with probability one when it is above $b^*$ under the schedule $P(b)$. For $b > b^*$ under the schedule $P(b)$, $b(+1) > b$ even under the best realization of the surplus. For a given pair $(b, b^*)$, the fixed point (5) need not be unique. In this context, nonuniqueness of the fixed point implies that the limit of the iteration

$$P_{n+1} = F\left(b, b^*, P_n\right)$$

depends on $P_0$. If we assume investors solve the model this way, then this means that the rational expectations equilibrium is dependent on their initial perceived probability of default. In this case, the equilibrium value of $b^*$ will also depend on the starting point. Throughout the paper, we will study the two extreme cases. Under the good equilibrium, the initial perceived probability of default is $P_0 = 0$ for all $b$; under the bad equilibrium, it is $P_0 = 1$ for all $b$.[4]

**Remark 1.** *The equilibrium probability of default $P^*(b)$ is a one-period hazard, the probability that the next period's debt lands in the region where $P = 1$. When all debt is one-period, this object is the probability of default. However, it need not be the case when there is debt with longer maturities. In this case, the probability of default is weakly larger for $b < b^*$: it includes the probability of entering an unsustainable path before the bond matures.*

Crucial to the nature of the equilibrium is whether $F'$, evaluated at some value of $b$ and $P^e$, is greater or smaller than one. If it is greater than one, it implies that the effect of an increase in $P^e$ has a more than one-for-one effect on $P$, leading to jumps in the probability of default, and in turn in the interest rate. Note that $F'$ is a decreasing function of uncertainty, measured by $e$. In the limit, as uncertainty, measured by $e$, goes to zero (and we go back to the case of certainty), $F'$ is infinite. Conversely, as uncertainty increases, $F'$ decreases. This suggests that the nature of the equilibrium depends on the degree of uncertainty.

We now characterize these regimes.

- For low uncertainty, the equilibrium resembles that under certainty: A range in which there are two equilibria, a good one, in which the probability of default is zero, and a

[4]There are, of course, intermediate cases. However, we focus on the two extreme ones. If the good and bad equilibria are different, then there will be other equilibria as well. If they are equal, then there is no multiplicity at all. In this sense, we can say the rational expectations equilibrium in this economy is supermodular on the initial perceived probability of default, $P_0$.

bad one, in which the probability of default is one. The size of the range is a decreasing function of uncertainty.

- For medium uncertainty, there is a range in which there are two equilibria. Still, the probability of default in the good equilibrium may be greater than zero, and the probability of default in the bad equilibrium may be less than one. Thus, interest rates in that range may reflect either bad fundamentals, multiple equilibria, or both.

- For high uncertainty, the multiplicity disappears, and in the unique equilibrium, when debt exceeds a threshold, the probability of default increases smoothly from zero to one. Fundamentals determine the outcome.

We now derive those results in turn.

## 3.1 Low uncertainty

**Proposition 2.** *If $e$ is small enough, that is, if $e \leqslant e_G = \frac{\theta(1+r)}{2r+\theta(1+r)}\, s$, then the equilibrium has the same structure as the equilibrium under certainty. If investors are optimistic, the probability of default jumps from zero to one at $(s-e)/r$. If investors are pessimistic, the probability of default jumps from one to zero at $(s+e)/R$. The range of multiplicity is given by $[(s+e)/R, (s-e)/r]$. It is thus decreasing as uncertainty increases.*

With more uncertainty, the probability of a higher surplus increases, making pessimistic investors less sure that there will be default at a given debt level, and thus requiring a higher level of debt to be certain of default, a higher $b_1^B$. With more uncertainty, the probability of a lower surplus also increases, making optimistic investors less certain that there will be no default at a given debt level, and thus requiring a lower level of debt to be sure of no default, a lower $b_0^G$. So $b_1^B$ and $b_0^G$ are closer to each other.

In addition, low uncertainty means values of $F'$ above one, leading to jumps. The probability of default, if investors are optimistic, jumps from zero to one at $b_0^G$, so $b_1^G = b_0^G$. If investors are pessimistic, the probability of default jumps from one to zero at $b_1^B$, so $b_0^B = b_1^B$. The equilibrium is shown in Figure 2. As uncertainty increases, the range of multiplicity decreases.

Two remarks on this regime. First, the two effects just described are distinct. The narrowing of the range comes from the bounds of the support, which move the two certainty thresholds; the sensitivity of beliefs, $F'$, only determines that the jumps are full. It is $F'$, not

the bounds, that will do the work in the next two regimes. Second, the range narrows from both ends, but have different implications. The rise in $b_1^B$ makes the bad equilibrium harder to sustain. The fall in $b_0^G$ lowers the level of debt that is sustainable even under optimism. Uncertainty does not make debt safer.

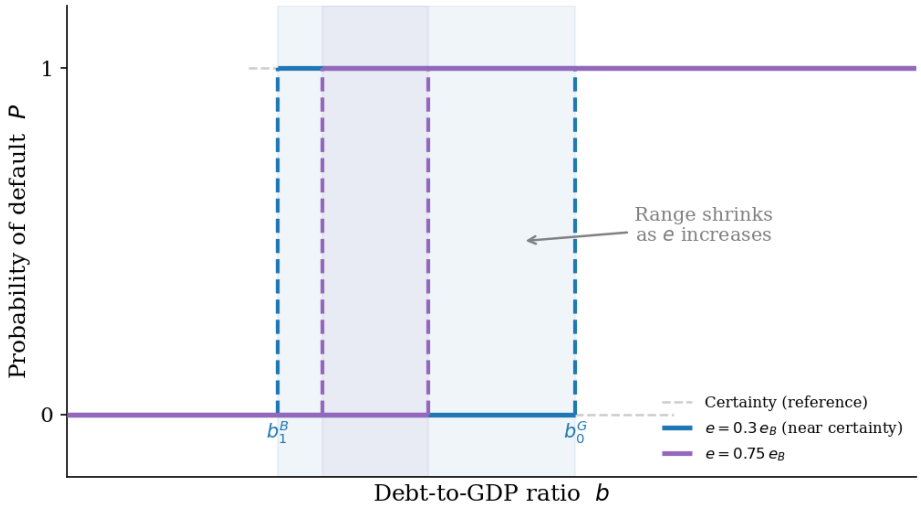



**Figure 2:** Multiple equilibria under low uncertainty.

The derivation is as follows:

Assume that the equilibrium has the same structure as the equilibrium under certainty, namely a jump from zero to one at some value $b_0^G$ (so that $b_1^G = b_0^G$) for the good equilibrium, and a jump from one to zero at some value $b_1^B$ (so that $b_0^B = b_1^B$) for the bad equilibrium.

Consider the bad equilibrium. If $b_1^B$ is the lowest value of debt consistent with $P = 1$, it must satisfy

$$F\left(b_1^B, b_1^B, 1\right) = 1,$$

so

$$b_1^B = \frac{s+e}{R}.$$

If the probability jumps from one to zero at $b_1^B$, so $b_0^B = b_1^B$, it must be that

$$F\left(b_1^B, b_1^B, 0\right) = 0,$$

which in turn implies that uncertainty must be such that

$$e \leqslant e_B = \frac{R-r}{R+r}\, s.$$

A condition that naturally holds is that, at $b_1^B$ and $P = 1$, $F'\left(b_1^B, b_1^B, 1\right) \geqslant 1$, so the probability indeed jumps from one to zero. For this condition to hold, we need

$$e \leqslant \bar{e} = \frac{\theta(1+r)}{2(1-\theta)(r+\theta) - \theta(1+r)} s.$$

One can verify that $\bar{e} \geqslant e_B$, so it is satisfied if $e \leqslant e_B$.

Intuition for the nature of the solution is given in Figure 3, which draws the relation between $P$ and $P^e$ at $b_1^B$, $P = F\left(b_1^B, b_1^B, P^e\right)$, and the 45-degree line. By definition of $b_1^B$, if $P^e = 1$, then $P = P^e = 1$. Consider now a small decrease in $P^e$. If $e$ is very small, $F'$ is very large, leading to a large decline in $P$, with two equilibria, $P = 1$ and $P = 0$. As $e$ increases, $F'$ decreases, the relation shifts to the left, and there are still two equilibria. The limit is reached when $e = e_B$, and the relation is given by the convex curve crossing the 45-degree line at both one and zero. Because $F$ is convex, it is clear that at $P = 1, F' > 1$, so that the equilibrium jumps from one to zero.

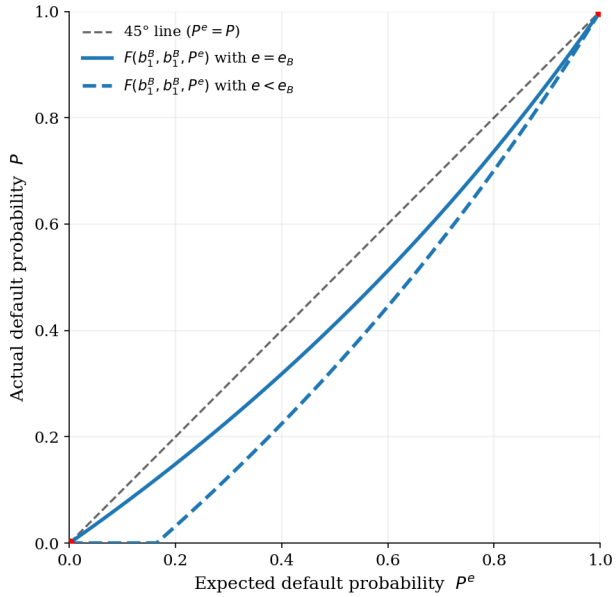



**Figure 3:** Bad equilibrium, low uncertainty

Consider now the good equilibrium. If $b_0^G$ is the highest value of debt consistent with

$P = 0$, it must satisfy

$$F\left(b_0^G, b_0^G, 0\right) = 0,$$

so

$$b_0^G = \frac{s - e}{r}.$$

For the probability to jump from zero to one at $b_0^G$, so $b_1^G = b_0^G$, it must be that

$$F\left(b_0^G, b_0^G, 1\right) = 1,$$

which in turn implies that uncertainty must be such that $e \leqslant e_B$.

One further condition must hold, however, that at $b_0^G$ and $P = 0$, $F'\left(b_0^G, b_0^G, 0\right) \geqslant 1$, so the probability indeed jumps from zero to one. This implies

$$e \leqslant e_G = \frac{\theta(1+r)}{2r + \theta(1+r)} s.$$

One can show that $e_G < e_B$, so $e_G$ is the relevant threshold for the equilibrium to have full jumps at $b_0^G = b_1^G$ and $b_1^B = b_0^B$.

Intuition for the nature of the solution (note that $F' \geqslant 1$ condition is now binding, whereas it was not for the bad equilibrium) is given in Figure 4.

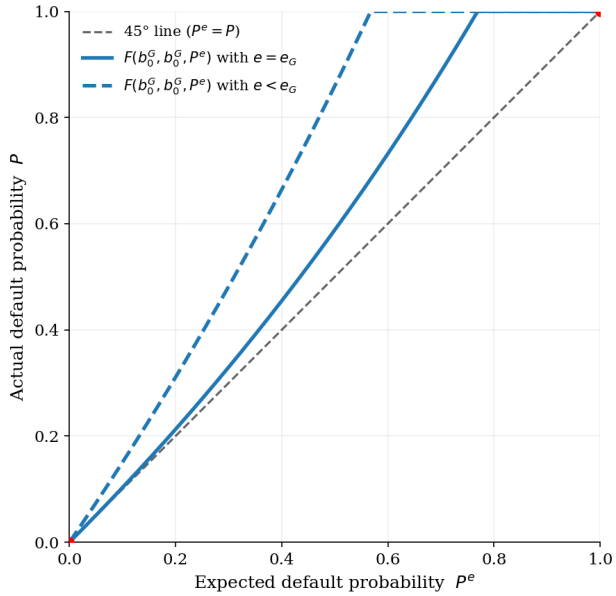



**Figure 4:** Good equilibrium, low uncertainty

Figure 4 draws the relation between $P$ and $P^e$ at $b = b_0^G$, namely $P = F\left(b_0^G, b_0^G, P^e\right)$ and the 45 degree line. By definition of $b_0^G$, if $P^e = 0$, then $P = P^e = 0$. Consider now a small increase in $P^e$. If $e$ is very small, $F'$ is very large, leading to a large increase in $P$, with two equilibria, $P = 1$ and $P = 0$. As $e$ increases, $F'$ decreases, the relation shifts to the right, and there are still two equilibria. The limit is reached when $e = e_G$, when $F'$ is equal to 1, and the function is tangent to the 45-degree line.

## 3.2 Medium uncertainty

**Proposition 3.** *If uncertainty is not too large, that is* $e < \bar{e}$, *where* $\bar{e} = \frac{\theta(1+r)}{2(1-\theta)(r+\theta)-\theta(1+r)}s$, *then, if investors are pessimistic and* $e_B < e$, *at* $b_1^B = (s+e)/R$, *the probability of default jumps down from one to* $P_B^*$ *given by* $P_B^* = F\left(b_1^B, b_1^B, P_B^*\right)$, *and then decreases smoothly with debt, until the probability of default is equal to zero.* $b_0^B$ *is given by* $b_0^B = \left(b_1^B + s - e\right)/(1+r)$.

*If* $e_G < e < \bar{e}$, *then, if investors are optimistic, the level of debt at which default takes place,* $b_1^G$, *and the probability jump from* $P^G$ *to one are jointly determined by the solution to the two equations,* $P_G^* = F\left(b_1^G, b_1^G, P_G^*\right)$ *and* $F'\left(b_1^G, b_1^G, P_G^*\right) = 1$. *The probability of default increases smoothly with* $b$ *from zero at* $b_0^G = \left(b_1^G + s - e\right)/(1+r)$ *to* $P_G^*$.

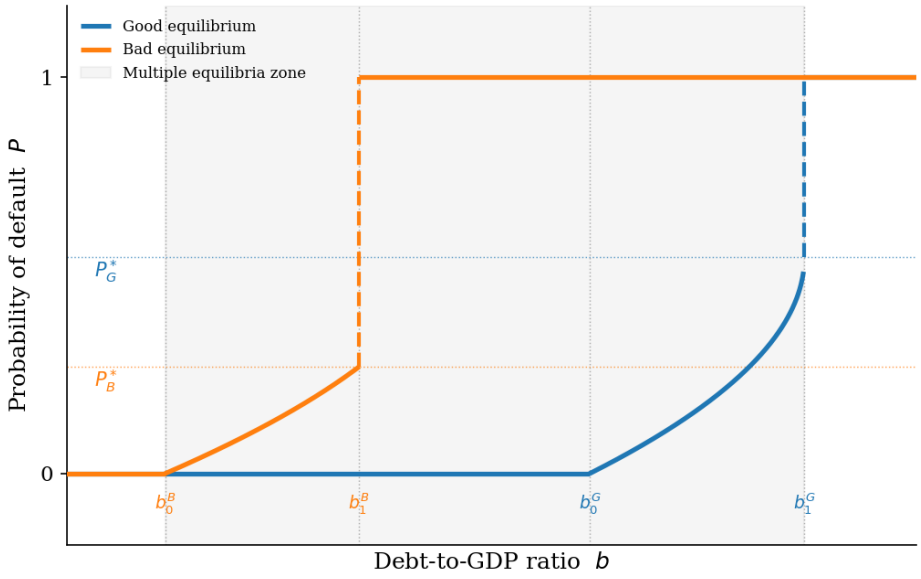



**Figure 5:** Multiple equilibria under medium uncertainty.

The equilibrium, for $e_B < e < \bar{e}$ is shown in Figure 5. If investors are optimistic, starting

at $b_0^G$, the probability of default, and thus the interest rate, increases smoothly as debt increases, and at $b_1^G$ and $P = P_G^*$, jumps to one. If investors are pessimistic, starting at $b_1^B$, the probability of default, and thus the interest rate, jumps down from one to $P_B^*$, and then decreases smoothly as debt decreases, until it reaches zero at $b_0^B$. [5]

The derivation is the following:

Consider the bad equilibrium. By the same argument as before, $b_1^B$ is still given by

$$F\left(b_1^B, b_1^B, 1\right) = 1,$$

so

$$b_1^B = \frac{s+e}{R},$$

and $F'\left(b_1^B, b_1^B, 1\right) > 1$. But now, $F\left(b_1^B, b_1^B, 0\right) > 0$. That is, even if $P^e = 0$, $P > 0$. This implies that there is only a partial jump of $P$ to $P_B^* > 0$, given by the smaller root of:

$$F\left(b_1^B, b_1^B, P_B^*\right) = P_B^*. \tag{6}$$

At $P_B^*$, $F'\left(b_1^B, b_1^B, P_B^*\right) < 1$, so for $b < b_1^B$, the probability $P$ adjusts smoothly downward with $b$ according to the implicit equation

$$F\left(b, b_1^B, P\right) = P.$$

The probability reaches $P = 0$ at $b_0^B$ defined by

$$F\left(b_0^B, b_1^B, 0\right) = 0,$$

which gives

$$b_0^B = \frac{b_1^B + (s-e)}{1+r}. \tag{7}$$

At some value of $e$, the solution for $P_B^*$ is equal to $P_B^* = 1$, and there is no longer a jump. This value is given by : $F'\left(b_1^B, b_1^B, 1\right) = 1$. Solving for this value shows that it is equal to $\bar{e}$

[5] There is an intermediate subcase where $e_G < e < e_B$. In that case, the good equilibrium has a smooth adjustment from 0 to $P_G^* > 0$ and then a jump to 1, but the bad equilibrium has a full jump from one to zero. The reason why $e_G < e_B$ is that the belief feedback is stronger near the bad equilibrium, due to the convexity of the pricing function, $F' \propto (1 - \theta P^e)^{-2}$. A change in belief moves the actual default probability much more when these beliefs are pessimistic.

derived earlier.

Intuition for the nature of the solution is given in Figure 6, which draws the relation between $P$ and $P^e$ given $b = b_1^B$, namely $P = F(b_1^B, b_1^B, P^e)$ and the 45-degree line.

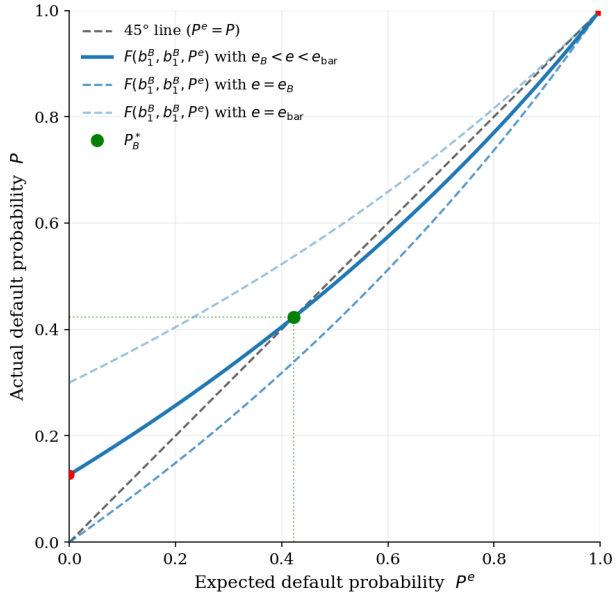



**Figure 6:** Bad equilibrium. Medium uncertainty

The locus to the right corresponds to the case where $e = e_B$ and the function $F$ intersects the 45-degree line at $P = 0$ and $P = 1$. As $e$ increases further, the function crosses the 45-degree line at $P_B^* > 0$, implying a jump from 1 to $P_B^*$. As $F$ is convex, at $P_B^*$, $F'\left(b_1^B, b_1^B, P_B^*\right) < 1$, so the adjustment of $P$ from $P_B^*$ to 0 is smooth until $P = 0$.

As $e$ increases further, at some point the function $F$ becomes tangent to the 45-degree line. There is no longer a jump, and $P$ decreases smoothly with debt. The value at which this happens is $\bar{e}$. Higher values correspond to the case of high uncertainty below.

Consider now the good equilibrium.

When $e > e_G$, two conditions must hold, which determine $b_1^G$. The first is that there is some probability $0 \leqslant P_G^* \leqslant 1$ such that $F\left(b_1^G, b_1^G, P_G^*\right) = P_G^*$. Second, that at $P_G^*$, $F'\left(b_1^G, b_1^G, P_G^*\right) = 1$, so the adjustment of $P$ between 0 and $P_G^*$ is smooth. Thus, in this case, $b_1^G$ and $P_G^*$ are jointly determined by the two equations above. For $e = e_G$, $P_G^* = 0$. For $e = \bar{e}, P_G^* = 1$

For $b < b_1^G$, the probability $P$ adjusts smoothly with $b$ according to the smaller root of

the equation

$$F\left(b, b_1^G, P\right) = P.$$

This traces out a continuous function $P(b)$ as $b$ decreases from $b_1^G$. The probability reaches $P = 0$ at the level $b_0^G$, defined by

$$F\left(b_0^G, b_1^G, 0\right) = 0,$$

which implies

$$b_0^G = \frac{b_1^G + (s - e)}{1 + r}. \tag{8}$$

Intuition for the nature of the solution is given by Figure 7. The function $F$ must be such that it is tangent to the 45-degree line. At that point, $F' = 1$ and $P = P_G^*$. As $e$ increases, the tangency point increases until, at $e = \bar{e}$, it is tangent to the 45-degree line at $P = P_G^* = 1$, and there is no longer a jump. The value of $b_1^G$ is then $(s + e)/R$, the same as $b_1^B$.

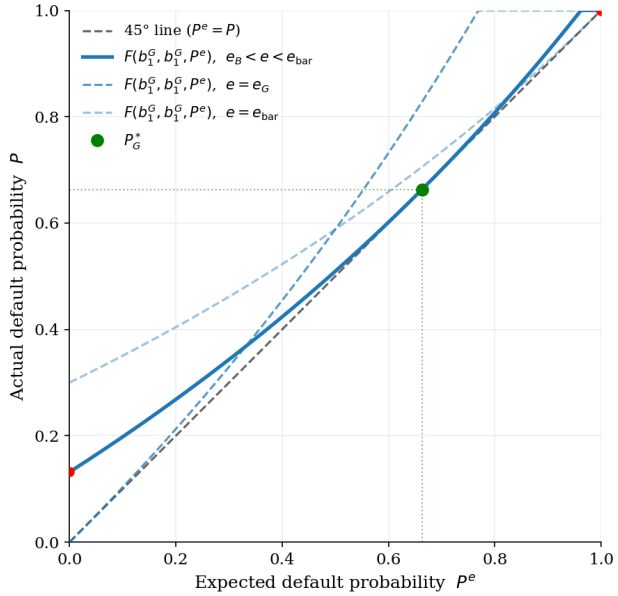



**Figure 7:** Good equilibrium. Medium uncertainty

## 3.3 High Uncertainty

**Proposition 4.** *When uncertainty is large, that is $e \geqslant \bar{e}$, the two loci coincide, and the equilibrium is unique: $b_1 \equiv b_1^B = b_1^G = (s + e)/R$ and $b_0 \equiv b_0^B = b_0^G = (b_1 + s - e)/(1 + r)$.*

*The probability of default is a continuous function of the level of debt, increasing from 0 for $b_0$ to 1 for $b_1$. Multiplicity disappears because, with high uncertainty, the effect of $P^e$ on $P$ is too weak to generate multiple equilibria. This regime exists within the admissible range $e < s$ if and only if $\bar{e} < s$, that is, if and only if $r(1-2\theta) > \theta^2$. For $r = 0.05$, this requires $\theta < 0.18$.*[6]

The equilibrium is represented by the graph in Figure 8

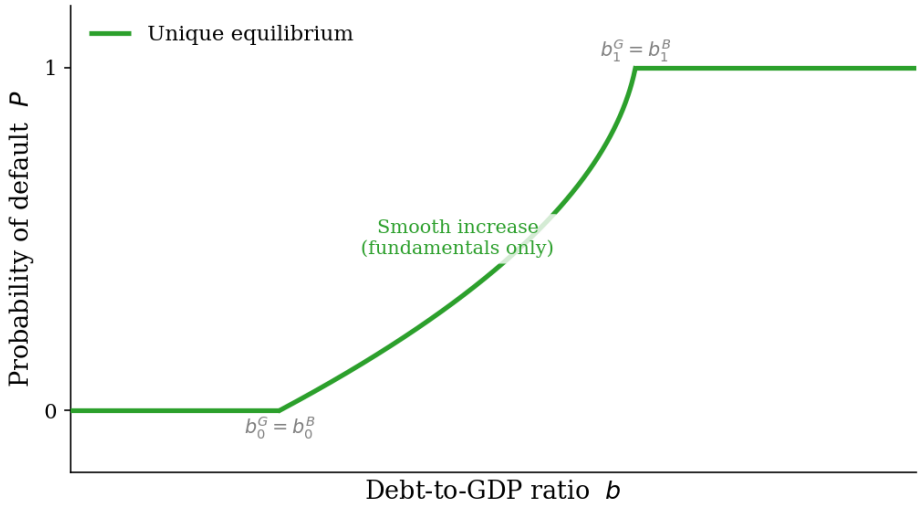



**Figure 8:** Equilibrium. High uncertainty

When $e \geqslant \bar{e}$, the good and bad equilibria collapse into a unique equilibrium. $b_1$ is given by $F(b_1, b_1, 1) = 1$, which implies $b_1 = (s+e)/R$.

At $b_1$, $F'(b_1, b_1, 1) \leqslant 1$, so the adjustment of the probability of default to debt is smooth.

The lower endpoint is given by $F(b_0, b_1, 0) = 0$, which implies $b_0 = (b_1 + s - e)/(1+r)$. $P$ increases continuously with debt, from zero at $b_0$ to one at $b_1$.

Intuition for the solution is given in Figure 9, which plots $P$ against $P^e$ when $b = b_1$. The tangent to the 45-degree line corresponds to $e = \bar{e}$, the boundary between the cases of medium and high uncertainty. Higher values of $e$ lead to a flatter relation and thus a unique equilibrium.

[6] When $r(1-2\theta) \leqslant \theta^2$, $F'\left(b_1^B, b_1^B, 1\right)$ remains above one for every $e < s$, and multiplicity persists at some levels of debt whatever the degree of uncertainty. The same condition guarantees that the denominator of $\bar{e}$ is positive. The condition is a joint restriction on the haircut and the safe rate: a larger haircut strengthens the feedback from beliefs to outcomes, and more uncertainty is needed to overcome it. Our calibration, $\theta = 0.1$ and $r = 0.05$ over a five-year period, satisfies it with some margin.

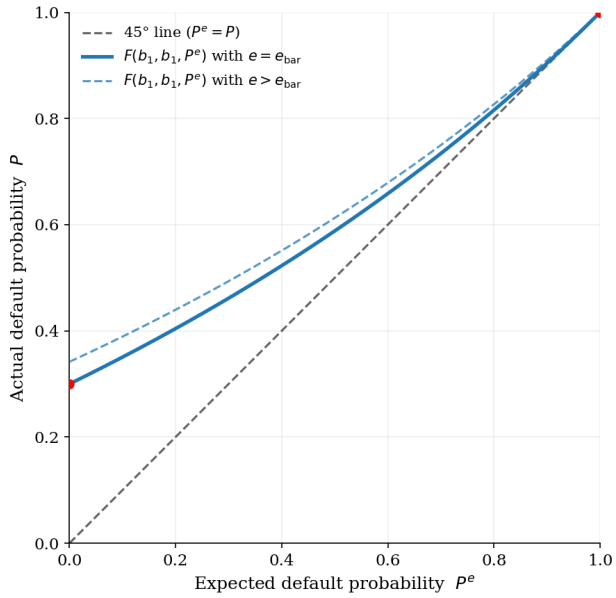



**Figure 9:** Equilibrium. High uncertainty

The three regimes are summarized in Figure 10, which maps them in the $(\theta, e/s)$ plane. The three boundaries $e_G$, $e_B$. and $\bar{e}$ are all proportional to $s$ and increasing in $\theta$. To the right of $\theta \approx 0.18$, $\bar{e}$ exceeds $s$ and the unique-equilibrium regime is empty.

## 3.4 Range of Multiplicity

We can now put the three cases together and show the effect of uncertainty on the range of multiple equilibria. There are two natural ways to measure it, and they answer different questions.

The first is the support of multiplicity, the set of debt levels over which the two schedules differ. For $e < \bar{e}$ it is the interval $\left[b_0^B, b_1^G\right]$, and its width is $b_1^G - b_0^B$. For $e \geqslant \bar{e}$ the two schedules coincide and the set is empty, even though $b_1 - b_0$ remains positive: it then measures the width of the unique, continuous transition from no default to default, not a range of multiplicity. The black line in the left panel of Figure 11 shows this width. It decreases at first, reaches a minimum within the medium-uncertainty regime, and increases again as $e$ approaches $\bar{e}$. The upturn is a genuine widening of the set of debt levels at which the two equilibria differ. As the jumps become partial, the two overlap over more, but by smaller and smaller amounts. Therefore, the width overstates multiplicity close to $\bar{e}$, where the two schedules differ by little over a long interval. Two corrections address this.

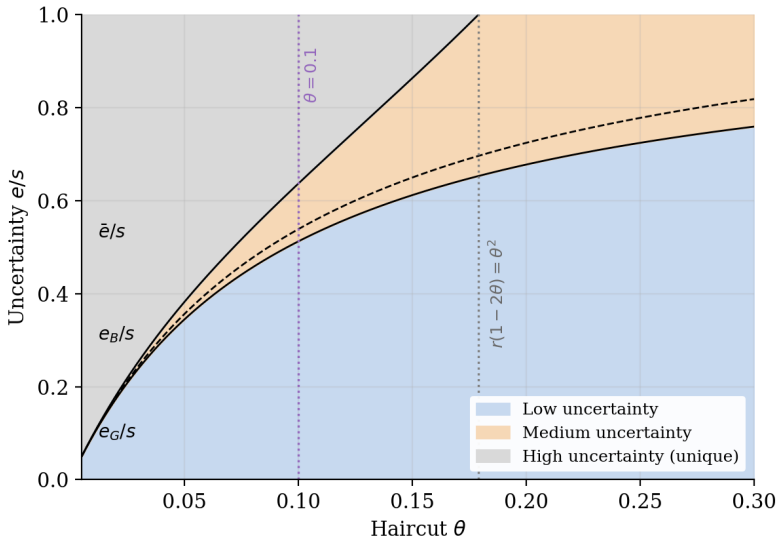



**Figure 10:** Regimes in the $(\theta, e/s)$ plane. The three boundaries are $e_G/s$ (lower solid line), $e_B/s$ (dashed line), and $\bar{e}/s$ (upper solid line). Low uncertainty: full jumps in both equilibria. Medium uncertainty: multiplicity with partial jumps; between $e_G$ and $e_B$ only the good equilibrium adjusts smoothly. High uncertainty: unique equilibrium. The grey dotted line marks $r(1-2\theta) = \theta^2$, beyond which $\bar{e} > s$ and the unique-equilibrium regime does not exist for $e < s$. The purple dotted line is our calibration, $\theta = 0.1$. Parameter: $r = 0.05$.

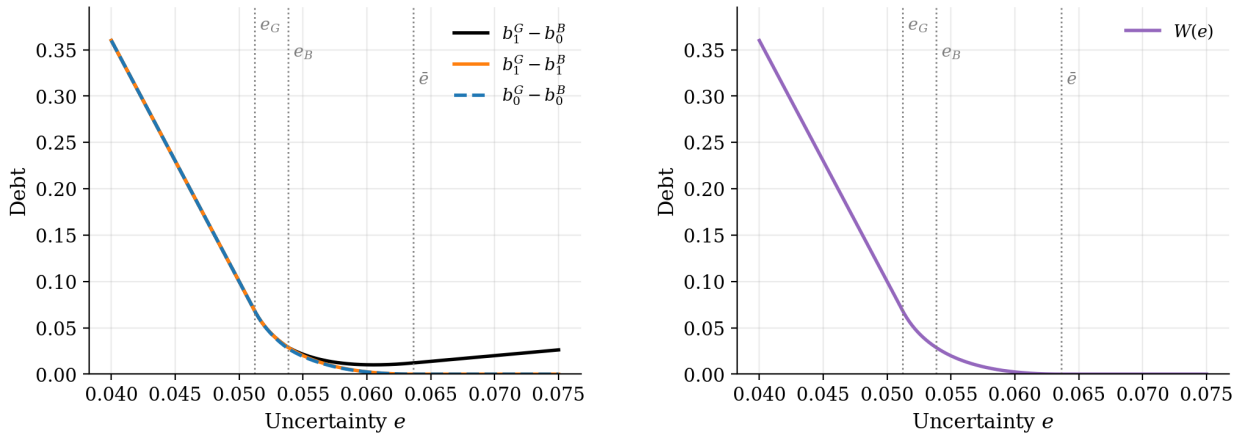
The first correction is to look at the thresholds themselves: the difference between $b_1^G$ and $b_1^B$, and the difference between $b_0^G$ and $b_0^B$. When there is no multiplicity, both are zero. With multiplicity, both decrease with $e$. This is shown in the left panel of Figure 11.

The second correction is to weight each level of debt by how much the two equilibria differ there, that is, to use the distance between the two equilibrium probabilities of default under the good and bad equilibrium. The largest difference between them, $\sup_b \left|P^B(b) - P^G(b)\right|$, is not informative: it is equal to one as long as there is a level of debt at which $P^B = 1$ and $P^G = 0$, which in our calibration is the case for all $e$ below 0.0583, well into the region of medium uncertainty. Instead, we use the Wasserstein distance, that is, the area between the two schedules:

$$W(e) := \int_0^\infty \left[P^B(b) - P^G(b)\right] db = \int_0^1 \left[b^G(p) - b^B(p)\right] dp, \tag{9}$$

where $b^i(p)$ denotes the lowest level of debt at which the probability of default under equi-

librium $i \in \{B, G\}$ reaches $p$.[7] The second expression gives $W$ a direct interpretation: it is the additional debt that optimistic investors tolerate for a given probability of default, averaged over all probabilities. Under low uncertainty both schedules jump from zero to one, and $W = b_1^G - b_1^B$. Under high uncertainty the two schedules coincide, and $W = 0$. The right panel of Figure 11 shows that $W$ decreases monotonically with $e$ and goes to zero as $e$ approaches $\bar{e}$.[8] Reporting the width and $W$ together gives the complete picture. As $e$ approaches $\bar{e}$, the set of debt levels over which the two equilibria differ does not shrink, but the two equilibria differ by less and less over it, until they coincide.




**(a)** Threshold-based measures

**(b)** Distance between the two schedules

**Figure 11:** Uncertainty and the range of multiplicity. Left: $b_1^G - b_0^B$ (black), $b_1^G - b_1^B$ (orange), and $b_0^G - b_0^B$ (blue, dashed). The first has an interior minimum and increases beyond $\bar{e}$; the two threshold gaps decrease with $e$ and vanish at $\bar{e}$. Right: Wasserstein distance, $W$, between the good and bad schedules, equation (9). Dotted vertical lines mark $e_G$, $e_B$, and $\bar{e}$. Parameters: $\theta = 0.1$, $r = 0.05$, $s = 0.1$.

[7] The integrand is nonnegative because the probability of default is never lower under the bad equilibrium. Since $P^G$ and $P^B$ are nondecreasing in $b$ and go from zero to one, they can be read as cumulative distribution functions, and $W$ is the Wasserstein distance between them; the second equality is the quantile representation of that distance. Lemma 1 in Appendix A characterizes $b^G(p)$ and $b^B(p)$.

[8] Other distances deliver the same result. This is the case of the $L^2$ distance between the two schedules, and of the integral over $b$ of the mutual information between the equilibrium being played, when the good and bad equilibria are equally likely, and the default event. We have verified that these three distances decrease monotonically with $e$ for several values of $\theta$ and $r$ for which $\bar{e} < s$.

# 4 Introducing a Feedback Rule

So far, we have assumed that primary balances were exogenous. It is more realistic to assume that, in response to higher debt, governments try to improve their finances. The properties of such a feedback rule were analyzed by Bohn (1998), who, however, focused on the good equilibrium, ignoring multiplicity. This section characterizes the good and bad equilibria under such a rule. The next section introduces uncertainty in the feedback rule itself.

Assume a feedback rule of the form:

$$s = \bar{s} + \epsilon + \alpha b,$$

where $\alpha$ is the feedback coefficient. The interesting (and realistic) assumption is first that $r < \alpha < R$. A rule with $\alpha < r$ would not lead to debt sustainability, and the analysis would be similar to the previous case, but with $r - \alpha$ instead of $r$. A rule with $\alpha > R$ (and $\alpha < 2 + R$) gives a stable process for debt, so $P = 1$ is never a fixed point and the equilibrium is unique with $P = 0$ for all $b$.[9] The rest of the model is the same as before. Debt dynamics are therefore given by:

$$b(+1) = \left( \frac{1 + r}{1 - \theta P^e} - \alpha \right) b - (\bar{s} + \epsilon).$$

**Proposition 5.** *Assume that the government follows a feedback rule, making the primary balance an increasing function of debt, so the primary balance is given by $\bar{s} + \epsilon + \alpha b$. Assume that $r < \alpha < R$. Then, under optimistic investors, debt is never expected to explode, and there is never default.*[10] *Under pessimistic investors, the probability of default jumps from one to zero at $b_1^B = (\bar{s} + e)/(R - \alpha)$. The jump is always from one to zero, independent of the degree of uncertainty. The feedback rule increases the level of debt at which multiplicity appears, but does not eliminate it.*

The equilibrium is represented in Figure 12. There is always a good equilibrium, with

[9] This result is also present in Lorenzoni and Werning (2019), where a sufficiently strong feedback rule restores equilibrium uniqueness. Stability of debt under optimism requires $|1 + r - \alpha| < 1$, that is $\alpha < 2 + r$, and not only $\alpha > r$. Since $\alpha < R$, this holds whenever $R < 2 + r$, or $\theta < 2/(3 + r)$, which is the case for any plausible haircut. We maintain this restriction throughout.

[10] Bohn (1998) studies the conditions under which debt under the good equilibrium has a finite present value. This only requires $\alpha > 0$. Indeed, when $0 < \alpha < r$ and there is no default, $b(+t) = (1 + r - \alpha)^t \left( b_0 - \frac{\bar{s}}{r - \alpha} \right) + \frac{\bar{s}}{r - \alpha}$ diverges for large $b_0$, yet $b(+t)/(1 + r)^t \to 0$. The debt ratio explodes while its discounted value vanishes. In our economy, default is triggered by an explosion of $b$, so $\alpha > r$ is the relevant condition.

$P = 0$. The debt threshold at which a bad equilibrium, and thus multiplicity, appears, increases with $\alpha$. The range of multiplicity is $\left[b_1^B, \infty\right)$, so it is unbounded above. In Section 5 we show that a cap on the fiscal effort restores a finite range of multiplicity.

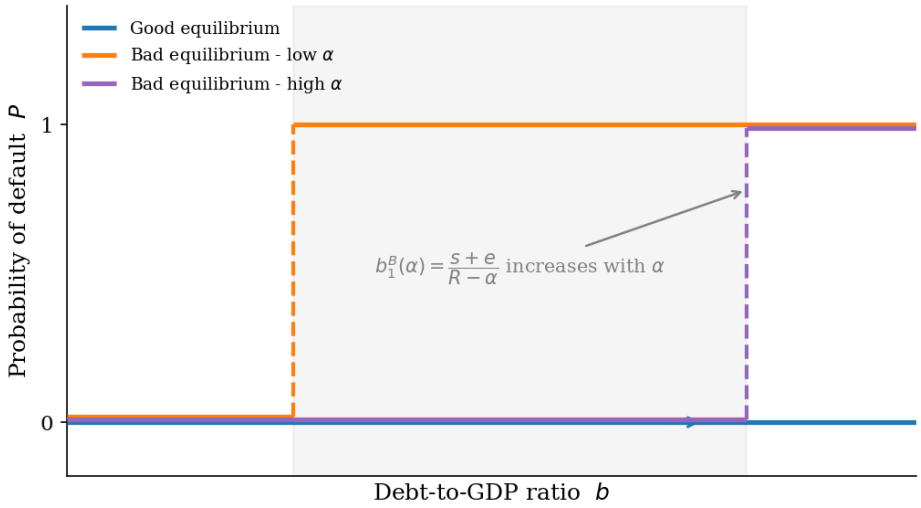



**Figure 12:** Multiple equilibria under the Bohn rule

The derivation is as follows (the steps parallel those of the exogenous primary balance case, so they are described in less detail than earlier).

If investors are optimistic, debt dynamics are given by:

$$b(+1) = (1 + r - \alpha)b - (s + e).$$

Under the assumption that $\alpha > r$, this is a stable $AR(1)$ process, so that the probability of a debt explosion, and thus of default, is equal to zero.

If investors are pessimistic, however, debt dynamics are given by:

$$b(+1) = (1 + R - \alpha)b - (s + \epsilon).$$

By assumption, $\alpha < R$, so $1 + R - \alpha > 1$. By the same argument as before, the bad equilibrium, i.e., the level of debt above which the probability of default is equal to one, is given by

$$b_1^B = \frac{s + e}{R - \alpha}.$$

Define the function $F$ in the same way as before, namely that, given $b$ today, a target

$b^*$, and a perceived probability $P^e$, it gives the probability that $b(+1) \geqslant b^*$. Then, to see whether the adjustment is fully discontinuous, i.e, whether $b_0^B = b_1^B$, we need two conditions:

The first is given by:

$$F(b_1^B, b_1^B, 0) = 0.$$

This condition can be rewritten as $(r - \alpha)b_1^B \leqslant (s - e)$, which is always satisfied as, by assumption, $e < s$, and $r - \alpha < 0$.

The second condition is given by:

$$F'(b_1^B, b_1^B, 1) \geqslant 1.$$

This condition can be rewritten as $\frac{\theta}{2e}\frac{1+R}{1-\theta}\frac{s+e}{R-\alpha} \geqslant 1$. To see whether it holds, evaluate it at the highest value of $e$, $e = s$. Use $\theta(1 + R) = (R - r)$ to rewrite the left hand side as: $\frac{1}{1-\theta}\frac{R-r}{R-\alpha}$ . Both terms are greater than 1. So $F' > 1$ is satisfied for any value of $e \leqslant s$.

Thus, the bad equilibrium is always discontinuous. The higher $\alpha$, the higher the threshold and the zone of multiplicity.

## 4.1 Multiplicity under an uncertain feedback rule

Investors may well question the commitment of the government to implementing a feedback rule strong enough to rule out default in the good equilibrium. Thus, in this section, we explore the implications of an uncertain feedback rule.

A natural way to think about this uncertainty is in terms of a fiscal adjustment program of limited duration. The government announces a consolidation that ties the primary balance to the level of debt, but the strength of the adjustment is uncertain: political resistance, implementation problems or unexpected developments may make the actual response of the primary balance stronger or weaker than announced. Once the program is over, adjustment fatigue sets in and the primary balance returns to its level without feedback. The question for investors is then whether the program will leave debt at a level that is sustainable without further adjustment.

We formalize this as follows. In the current period, the primary balance is

$$s = \bar{s} + (\alpha + \eta)\, b, \qquad \eta \sim U\left[-a, a\right],$$

so that debt dynamics are given by

$$b(+1) = \left( \frac{1+r}{1-\theta P^e} - \alpha - \eta \right) b - \bar{s}.$$

A realization $\eta = a$ implies the strongest adjustment; a realization $\eta = -a$ the weakest. To focus on uncertainty about the feedback coefficient, we set $e$ equal to zero. From the next period onwards, there is no feedback and no uncertainty: the primary balance is equal to $\bar{s}$, and the economy is the one studied in Section 2. We assume that investors are optimistic in that continuation.[11] Debt next period is then sustainable if $b(+1) \leqslant \bar{s}/r$, while for $b(+1) > \bar{s}/r$ debt increases forever even at the safe rate, and default is certain. Default on the debt issued today therefore takes place if and only if $b(+1) > c \equiv \bar{s}/r$.[12]

The key difference with the previous sections is that the threshold $c$ does not depend on the equilibrium selected in the current period. The probability of default is

$$F\left(b, c, P^e\right) = Prob\left(b(+1) > c\right) = clip_{[0,1]} \left\{ \frac{1}{2ab} \left[ \left( \frac{1+r}{1-\theta P^e} - \alpha + a \right) b - \bar{s} - c \right] \right\}, \quad (10)$$

and its derivative with respect to $P^e$,

$$F'\left(b, c, P^e\right) = \frac{\theta(1+r)}{2a\left(1-\theta P^e\right)^2},$$

does not depend on $b$ and, as before, decreases with uncertainty. It is convenient to define $K \equiv \bar{s} + c = \bar{s}(1+r)/r$ and

$$g(P) \equiv \frac{1+r}{1-\theta P} - \alpha + a - 2aP.$$

Then $F(b, c, P) - P$ has the sign of $g(P)\, b - K$: an interior fixed point at debt $b$ satisfies

[11] If investors are also pessimistic in the continuation, default under the bad equilibrium occurs as soon as $b(+1) > \bar{s}/R$. The analysis below goes through, with $c = \bar{s}/R$ for the bad equilibrium, but multiplicity then never disappears, because it is already present in the continuation: in our calibration, $b_1^G - b_1^B$ remains above 1.3 for all the values of $a$ in Figure 13. Assuming optimism in the continuation isolates the multiplicity that arises from beliefs about the adjustment program itself.

[12] If the uncertain coefficient were instead drawn anew every period, then, since $\alpha > r$, debt would decrease with probability larger than one half in every period. Under the safe rate, debt would be retired with probability one, because, provided $a < 1 + r - \alpha$ so that the multiplier $1 + r - \alpha - \eta$ is positive for every realization, $\mathbb{E}\left[\log\left(1 + r - \alpha - \eta\right)\right] \leqslant \log\left(1 + r - \alpha\right) < 0$, and the schedule $P(b) = 0$ would be an equilibrium at every level of debt, whatever the degree of uncertainty. Limiting the uncertain adjustment to one period avoids this by pinning down a finite default region through the continuation.

$g(P) = K/b$, $P = 0$ is a fixed point if and only if $g(0) \leqslant K/b$, and $P = 1$ is a fixed point if and only if $g(1) \geqslant K/b$. The function $g$ is convex, with $g'(P) = 2a\,(F' - 1)$, so its shape is governed by the same comparison between $F'$ and one as in Section 3.

**Proposition 6.** *Let $a_G = \theta(1+r)/2$, $a_B = (R-r)/2$ and $\bar{a} = \frac{\theta(1+r)}{2(1-\theta)^2}$, so that $a_G < a_B < \bar{a}$.*

1. Low uncertainty. *If investors are optimistic and $a \leqslant a_G$, the probability of default jumps from zero to one at $b_0^G = b_1^G = K/(1+r-\alpha+a)$. If investors are pessimistic and $a \leqslant a_B$, the probability of default jumps from one to zero at $b_0^B = b_1^B = K/(1+R-\alpha-a)$.*

2. Medium uncertainty. *If investors are optimistic and $a_G < a < \bar{a}$, the probability of default increases smoothly from zero at $b_0^G = K/(1+r-\alpha+a)$ to $P_G^* = \frac{1}{\theta}\left(1 - \sqrt{\frac{\theta(1+r)}{2a}}\right)$, and jumps to one at*

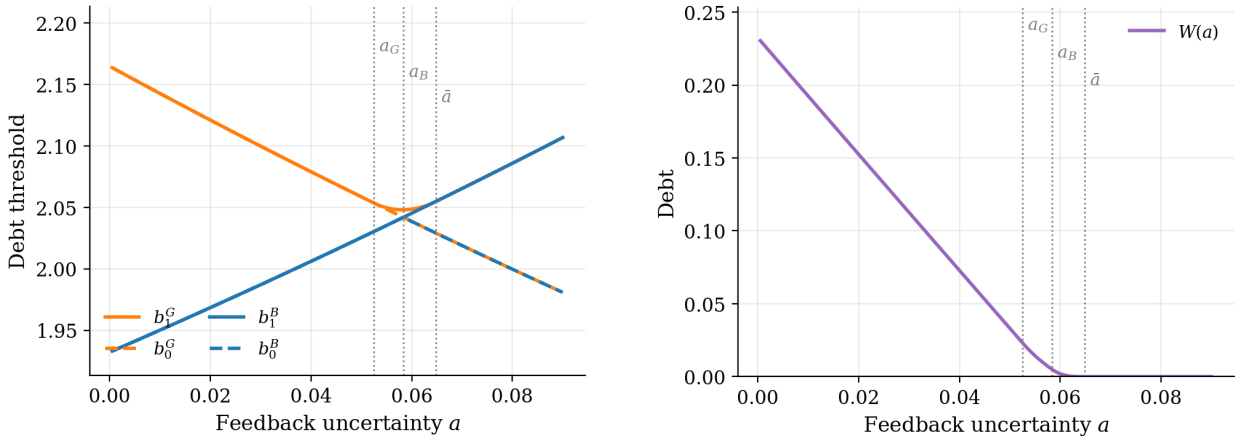
$$b_1^G = \frac{K}{g\left(P_G^*\right)} = \frac{K}{2\sqrt{2a(1+r)/\theta} - \alpha + a - 2a/\theta}.$$

   *If investors are pessimistic and $a_B < a < \bar{a}$, the probability of default jumps from 1 to $P_B^* = \frac{1}{\theta}\left(1 - \frac{\theta(1+r)}{2a(1-\theta)}\right)$ at $b_1^B = K/(1+R-\alpha-a)$, and then decreases smoothly with debt until it reaches zero at $b_0^B = K/(1+r-\alpha+a) = b_0^G$.*

3. High uncertainty. *If $a \geqslant \bar{a}$, the equilibrium is unique: $b_1 \equiv b_1^G = b_1^B = K/(1+R-\alpha-a)$, $b_0 \equiv b_0^G = b_0^B = K/(1+r-\alpha+a)$, and the probability of default increases smoothly from zero at $b_0$ to one at $b_1$.*

The structure of the equilibrium is the same as under an uncertain primary balance, and so is the intuition. Pessimistic investors are certain of default only if debt exceeds $c$ even under the strongest possible adjustment, so their threshold $b_1^B$ depends only on $\alpha + a$. Optimistic investors are certain that there is no default only if debt stays below $c$ even under the weakest adjustment, so their threshold $b_0^G$ depends only on $\alpha - a$. More uncertainty about the program thus raises $b_1^B$ and lowers $b_0^G$. For pessimistic investors, only the possibility of a stronger adjustment matters, since under a weaker one default would have occurred anyway, an intuition similar to the bad news principle in Bernanke (1983). In addition, larger uncertainty weakens the effect of the perceived probability of default on the actual one, as $F'$ decreases with $a$. As a result, multiplicity decreases with $a$, and it disappears when $a \geqslant \bar{a}$. Figure 13 shows the thresholds and the distance $W$ between the two schedules, defined as in (9), which decreases monotonically with $a$.

Three differences with the case of an uncertain primary balance are worth noting. First, as $a \to 0$, the thresholds converge to $K/(1+r-\alpha)$ and $K/(1+R-\alpha)$, and a stronger program, a higher $\alpha$, raises both; with $\alpha = 0$, the optimistic threshold is $\bar{s}/r$, as in Section 2. Second, the optimistic threshold is finite for any degree of uncertainty, including when $\alpha - a \geqslant r$, so that the program reduces debt under all realizations: a single period of adjustment cannot bring an arbitrarily large debt below $\bar{s}/r$. Third, since the default threshold $c$ is common to both equilibria, the lower thresholds $b_0^G$ and $b_0^B$ coincide as soon as $a > a_B$, and multiplicity then only concerns the upper part of the range.




**(a)** Debt thresholds

**(b)** Distance between the two schedules

**Figure 13:** Uncertain feedback and multiple equilibria. Left: thresholds of the good equilibrium, $b_1^G$ and $b_0^G$ (orange), and of the bad equilibrium, $b_1^B$ and $b_0^B$ (blue), as functions of the uncertainty $a$ about the feedback coefficient. Right: Wasserstein distance $W$ between the good and bad schedules. Dotted vertical lines mark $a_G$, $a_B$ and $\bar{a}$. Parameters: $\theta = 0.1$, $r = 0.05$, $\bar{s} = 0.1$, $\alpha = 0.08$.

The derivation uses the function $g$. Under optimistic investors, the probability of default is the smallest fixed point, that is, the smallest $P$ such that $g(P) \leqslant K/b$, or one if there is none. It is zero for $b \leqslant K/g(0)$ and one for $b > K/\min_{[0,1]} g$, so $b_0^G = K/g(0)$ and $b_1^G = K/\min_{[0,1]} g$. If $a \leqslant a_G$, then $F' \geqslant 1$ at $P = 0$, $g$ is increasing, and $b_1^G = b_0^G$. If $a_G < a < \bar{a}$, $g$ reaches its minimum at the interior point where $F' = 1$, which gives $P_G^*$; between $b_0^G$ and $b_1^G$ the probability of default moves along the decreasing part of $g$. If $a \geqslant \bar{a}$, then $F' \leqslant 1$ on $[0,1]$, $g$ is decreasing, and $b_1^G = K/g(1)$.

Under pessimistic investors, the probability of default is the largest fixed point, that is, the largest $P$ such that $g(P) \geqslant K/b$, or zero if there is none. It is one for $b \geqslant K/g(1)$, so $b_1^B = K/g(1)$. If $a \leqslant a_B$, then $g(0) \leqslant g(1)$ and, by convexity, $g < g(1)$ on $(0,1)$: below $b_1^B$ the only fixed point is zero. If $a_B < a < \bar{a}$, then $g(0) > g(1)$ and $g'(1) > 0$, so $g(P) = g(1)$

has a second solution $P_B^* \in (0,1)$, which gives the partial jump; the probability of default then decreases along the decreasing part of $g$ until it reaches zero at $K/g(0)$. If $a \geqslant \bar{a}$, $g$ is decreasing, the fixed point is unique at every level of debt, and the two equilibria coincide.

# 5 Extensions

Asymmetric Uncertainty Our result that higher uncertainty reduces multiplicity assumes increases in uncertainty are symmetric. In practice, they often come from larger uncertainty on the downside, the probability that the outlook for the primary balance is worse than expected. In this case, the lower expected primary balance also decreases the range of multiplicity. To see this, suppose fiscal surplus is equal to $s + \epsilon$, where $\epsilon$ follows a uniform distribution on $[-e_L, e_H]$ with $e_L \neq e_H$. Define

$$s' := s + \mathbb{E}[\epsilon] \qquad \epsilon' := \epsilon - \mathbb{E}[\epsilon].$$

Then, the fiscal surplus can be rewritten as $s' + \epsilon'$, where $\epsilon'$ has a uniform distribution over $[-e, e]$ with $e = (e_L + e_H)/2$. Larger uncertainty on the downside, an increase in $e_L$, has two effects, both of which decrease the range of multiplicity: $e$ increases and $s'$ decreases. The mechanism deserves to be spelled out. The threshold of the bad equilibrium, $b_1^B = (s + e_H)/R$, depends only on the best possible outcome and is unchanged. The threshold of the good equilibrium, $b_0^G = (s - e_L)/r$, depends only on the worst possible outcome and falls. The range narrows entirely because the region of certain default expands: downside uncertainty does not make the bad equilibrium harder to sustain, it makes the good one easier to lose. The formulas assume that the lowest possible surplus remains positive, $e_L < s$; for larger downside uncertainty the thresholds would have to be rederived.

Nonuniform Distribution of Fiscal Surpluses Our specification links the standard deviation to the bounds of the distribution. By changing $e$, we are shifting both the bounds and the standard deviation at the same time. To study its effects separately, we consider a truncated normal distribution between $s_L$ and $s_H$. By increasing the standard deviation $\sigma$ while keeping the truncation points fixed, we can increase uncertainty while keeping the bounds fixed. As $\sigma$ increases, the distribution converges to a uniform distribution. Our main result is that the effect of $\sigma$ is not monotone. Keeping $e$ fixed, a very concentrated distribution always produces multiplicity, whatever the value of $e$; a sufficiently dispersed one

eliminates it when $e > \bar{e}$ and not otherwise; and in between, an increase in $\sigma$ can go either way. What matters is the density of the surplus near the boundary between repayment and default, not just its variance.

More formally, assume the fiscal surplus has a truncated normal distribution with mean $\mu$ and variance $\sigma^2$, truncated between $s_L$ and $s_H$. The location parameter $\mu$ shapes the value around which fiscal surpluses are more frequent, and the variance $\sigma^2$ governs their dispersion. As $\sigma^2$ rises, extreme values become more frequent. When the range of surpluses is symmetric, $s_H = \mu + e$ and $s_L = \mu - e$. We maintain this assumption throughout, and we set $\mu = s$, so that the truncated normal has the same central value as the surplus in the certainty case.

Notice that as $\sigma \to \infty$, the distribution of surpluses converges to a uniform distribution, and we recover the three cases of low, medium, and high uncertainty of our baseline model. As $\sigma \to 0$, the density converges to a Dirac delta centered at $\mu$. For a finite value of $\sigma$, and given a threshold $b^*$, the probability of default is

$$P\left(s \leqslant \frac{1+r}{1-\theta P^e} b - b^*\right) = \Phi^t\left(\frac{1+r}{1-\theta P^e} b - b^*\right),$$

where $\Phi^t(\cdot)$ is the CDF of the truncated normal distribution. Under the bad equilibrium, $P = 1$, and

$$b_1^B = \frac{s+e}{R}.$$

Therefore, $b_1^B$ is unaffected by the distribution of fiscal surpluses. Two parameters shape the degree of uncertainty in this general case, the range $s_H - s_L = 2e$ and the variance $\sigma^2$. Both raise the variance of the surpluses.[13]

The properties of the good and bad equilibria depend on both $e$ and $\sigma$. Here we do not characterize this dependence in full, but rather show what is and is not similar to the effect of $e$: very low volatility always yields multiplicity, and enough volatility eliminates it when the support is wide enough, but the relation is not monotone in between. The derivative of

[13] When $s_H = \mu + e$ and $s_L = \mu - e$,

$$\mathrm{Var}(s) = \sigma^2 \left[1 + \frac{\frac{-e}{\sigma}\varphi\left(\frac{-e}{\sigma}\right) - \frac{e}{\sigma}\varphi\left(\frac{e}{\sigma}\right)}{\Phi\left(\frac{e}{\sigma}\right) - \Phi\left(\frac{-e}{\sigma}\right)} - \left(\frac{\varphi\left(\frac{-e}{\sigma}\right) - \varphi\left(\frac{e}{\sigma}\right)}{\Phi\left(\frac{e}{\sigma}\right) - \Phi\left(\frac{-e}{\sigma}\right)}\right)^2\right].$$

the probability of default with respect to $P^e$

$$F'\left(b, b^*, P^e\right) = \underbrace{\frac{2e}{\sigma} \cdot \frac{\varphi\left(\frac{\frac{1+r}{1-\theta P^e} b - b^* - s}{\sigma}\right)}{\Phi\left(\frac{e}{\sigma}\right) - \Phi\left(-\frac{e}{\sigma}\right)}}_{\Delta(e,\sigma,P^e,b)} \times F'_{uniform}\left(b, b^*, P^e\right), \tag{11}$$

where $F'_{uniform}$ denotes the derivative of $F$ assuming fiscal surpluses follow a uniform distribution, and $\varphi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and cdf of a standard normal distribution.

Consider first the case of a small value of $\sigma$. When $\sigma$ is small enough, the density is mostly concentrated around $s$. In this case, $F\left(b_1^B, b_1^B, P^e\right)$ resembles a step function of $P^e$, located at the belief for which debt next period, net of the mean surplus $s$, is exactly $b_1^B$, and the fixed point $P = F\left(b_1^B, b_1^B, P\right)$ has three solutions: one close to zero, another equal to one, and an intermediate value. Thus, there will be multiple equilibria.[14] Therefore, for low enough values of uncertainty, there will be multiple equilibria regardless of $e$, as long as $rb_1^B < s$, which in our calibration holds for all $e < s$ (see Proposition 9 in Appendix A).

As $\sigma$ increases, the density becomes flatter in the middle but higher at the edges. To see how this impacts the degree of multiplicity, evaluate (11) at $b = b_1^B$ and $P^e = 1$:

$$F'\left(b_1^B, b_1^B, 1\right) = \underbrace{\frac{2e}{\sigma} \cdot \frac{\varphi\left(\frac{e}{\sigma}\right)}{\Phi\left(\frac{e}{\sigma}\right) - \Phi\left(-\frac{e}{\sigma}\right)}}_{\Delta\left(\frac{e}{\sigma}\right)} \times F'_{uniform}\left(b_1^B, b_1^B, 1\right), \tag{12}$$

where $\Delta\left(e/\sigma\right) = \Delta(t) < 1$, with $\Delta'(t) < 0$, $\lim_{t\to 0} \Delta(t) = 1$ and $\lim_{t\to\infty} \Delta(t) = 0$ (Lemma 2 in Appendix A). Therefore, at this point, the sensitivity of the actual probability to the perceived probability is always smaller under the truncated normal than under the uniform distribution. A *necessary* condition for an equilibrium to be unique is that $F'\left(b_1^B, b_1^B, 1\right) \leqslant 1$. Under the uniform distribution, this condition is also *sufficient*. However, as seen in the case with a small value of $\sigma$, this need not be the case in a more general distribution. Proposition 8 in Appendix A gives a condition that is both necessary and sufficient.

When $e > \bar{e}$, $F'_{uniform} < 1$ regardless of the value of $\sigma$, and Lemma 2 implies that $F'$ at the top of the support is smaller still. This is not enough for uniqueness, however. Under a non-uniform distribution, the sensitivity of beliefs need not peak at the top of the support (see the remark after Proposition 8). For small $\sigma$, the mass concentrated around $s$ makes $F$

[14] That belief is $P^e = \frac{1}{\theta}\left[1 - \frac{(1+r)b_1^B}{b_1^B + s}\right]$, which lies in $(0,1)$ if and only if $rb_1^B < s$.

steep at an interior belief, and there are multiple equilibria even though $F' < 1$ at the edge. As $\sigma$ increases, this interior concentration dissolves and the equilibrium becomes unique. Proposition 9 establishes the two limits; in our calibration, the transition between them is monotone, with the equilibrium unique if and only if $\sigma$ is large enough (Appendix A.5).

When $e_B < e < \bar{e}$, there are multiple equilibria under a uniform distribution. In this case, a non uniform distribution discourages the existence of multiple equilibria, provided $\sigma$ is not too high. To see this, start from a very high value of $\sigma$ such that the distribution resembles a uniform. As we decrease $\sigma$, by (12) $F'$ decreases relative to $F'_{uniform}$. This operates *against* the existence of multiple equilibria, and, when $e$ is close enough to $\bar{e}$, there is uniqueness for intermediate values of $\sigma$ (Figure 14). However, when $\sigma$ is low enough, there is enough mass in the center to generate multiplicity of equilibria in the other way discussed above.

Our results allow us to decouple the effects of the *range* and the *volatility* of the surpluses. The range acts through the bounds of the support and through $F'$ at the top of the support, and its effect is monotone. Volatility acts through the density near the boundary between repayment and default, and its effect is not: for a given range, very low volatility always produces multiplicity, and when $e < \bar{e}$ so does high volatility, with uniqueness possible in between. The general lesson is that what matters for multiplicity is how uncertainty alters the density of outcomes near that boundary, not just dispersion.

Uncertainty about Future Equilibrium Selection In the baseline model, we derived our results under the assumption that investors' initial beliefs are constant over time. If investors select the good equilibrium in period $t$, they remain in the good equilibrium forever. In practice, however, investors face uncertainty about future market sentiment. In this section, we show that if default occurs when debt is explosive even under the best realization of the surplus and the most favorable future path of equilibrium selection, the main results of the baseline model carry over. The good equilibrium remains unchanged, while the possibility that markets become optimistic in the future weakly reduces the probability of default under the bad equilibrium. Aggregate uncertainty about the surplus remains the key determinant of multiplicity.

To see this, suppose investors' beliefs about equilibrium selection can change over time. At each date $t$, investors hold beliefs over the entire future path of equilibrium selection, and we do not restrict this path to follow a particular stochastic process. We extend the default rule of the baseline model as follows: investors refuse to refinance, and default occurs at settlement, only when next period's debt under full repayment would increase forever with

probability one even if the good equilibrium were selected in every subsequent period. As in the baseline model, this rule is imposed rather than derived from investors' decisions. The rule requires only that investors do not rule out the path along which the market remains optimistic forever. The probabilities they attach to different paths do not matter in this setting. In each period $t$, the equilibrium selected in that period determines the initial condition $P_{t,0}$ of the mapping

$$P_{t,n+1} = F\left(b, b^*, P_{t,n}\right),$$

where $b^*$ is the default threshold implied by this rule. By assumption, investors assign positive probability to the path along which the good equilibrium is selected forever. Therefore, if debt becomes so large that it is unsustainable even along this permanently optimistic path, default occurs with probability one. Otherwise, there exists a future path of equilibrium selection with positive probability under which debt does not explode. For this reason, the relevant default threshold is $b_1^G$ from the baseline model, and the probability of default is the probability that next period's debt exceeds $b_1^G$. Thus, in each period $t$, investors solve

$$P_{t,n+1} = F\left(b, b_1^G, P_{t,n}\right), \tag{13}$$

for either $P_{t,0} = 0$ or $P_{t,0} = 1$. This expression immediately implies that the probability of default under the good equilibrium remains unchanged. When $e \geqslant \bar{e}$, $b_1^G = b_1^B$, so (13) coincides with the baseline mapping, which has a unique fixed point. Therefore, there is no multiplicity in this economy either.

When $e < \bar{e}$, $b_1^G > b_1^B$ and there are multiple equilibria. The good equilibrium is the same as in the baseline model, but the bad equilibrium changes. Let $\tilde{b}_1^B$ denote the smallest level of debt at which default occurs with probability one under the bad equilibrium:

$$\tilde{b}_1^B = \min\left\{b : \ F\left(b, b_1^G, 1\right) = 1\right\} = \frac{b_1^G + s + e}{1 + R}.$$

Since $b_1^B < b_1^G$, it follows that $b_1^B < \tilde{b}_1^B < b_1^G$. For $b \in \left[\tilde{b}_1^B, b_1^G\right)$, the pessimistic interest rate pushes next period's debt above $b_1^G$ for every surplus realization, so default is certain under the bad equilibrium even though current debt is below the common boundary. For $b \in \left[b_1^B, \tilde{b}_1^B\right)$, the probability of default under the bad equilibrium is strictly lower than in the baseline model. It equals one under constant beliefs, whereas here it is strictly below one, because next period's debt can remain below $b_1^G$, where future optimism prevents explosion. For $b < b_1^B$, it is weakly lower, since raising the default boundary from $b_1^B$ to $b_1^G$ lowers $F$

pointwise.

As in the baseline model, the probability of default under the bad equilibrium jumps discretely at $\tilde{b}_1^B$ and varies continuously below it. Whether the jump is full or partial depends on whether the probability of default is positive just below $\tilde{b}_1^B$. The jump is full if and only if $F\left(\tilde{b}_1^B, b_1^G, 0\right) = 0$, which is equivalent to

$$2e \leqslant \theta\left(b_1^G + s + e\right).$$

Let $\tilde{e}_B$ denote the value of $e$ at which this condition holds with equality. When $\tilde{e}_B < e < \bar{e}$, the probability of default jumps from a value in $(0, 1)$ to one as debt crosses $\tilde{b}_1^B$. When $e \leqslant \tilde{e}_B$, it jumps from zero to one. Because the default boundary is now $b_1^G$ rather than $b_1^B$, $\tilde{e}_B$ differs from the corresponding threshold in the baseline model.

We have not explored the case in which investors assign zero probability to the market remaining optimistic forever. Since the relevant default threshold may then differ from $b_1^G$, we leave the analysis of this case for future research.

Capped Fiscal Effort The feedback rule of Section 4 assumes a country can increase the primary balance as debt increases. However, this assumption is unlikely to hold in reality for large debt levels. In this case, the properties of the multiple equilibria will resemble those of the baseline model. To show this, suppose instead there is a cap on the fiscal effort. The feedback rule is

$$s = \min\left\{\bar{s} + \alpha b, s_{max}\right\}.$$

Then, when $b > b_{max} = (s_{max} - \bar{s})/\alpha$, fiscal revenue does not react to an increase in debt. Under certainty, the debt threshold under the good equilibrium is

$$b_0^G = \frac{s_{max}}{r},$$

which is automatically larger than $b_{max}$ because $\alpha > r$. For the bad equilibrium, suppose $b_1^B \geqslant b_{max}$. Then surplus is frozen at $s_{\max}$ and

$$b_1^B = \frac{s_{\max}}{R},$$

which exceeds $b_{\max}$ as long as $\bar{s}/(R-\alpha) \geqslant s_{\max}/R$. Suppose instead $b_1^B \leqslant b_{\max}$. Then the feedback rule is still active and

$$b_1^B = \frac{\bar{s}}{R-\alpha},$$

which is below $b_{\max}$ under the condition $\bar{s}/(R-\alpha) \leqslant s_{\max}/R$. The two cases are exhaustive and meet at $\bar{s}/(R-\alpha) = s_{\max}/R$, where $b_1^B = b_{\max}$. Hence

$$b_1^B = \min\left\{\frac{s_{\max}}{R},\ \frac{\bar{s}}{R-\alpha}\right\}.$$

To introduce uncertainty, we assume the cap affects only the deterministic portion of the surplus, so that

$$s = \min\left\{\bar{s} + \alpha b,\ s_{\max}\right\} + \epsilon.$$

As in the certainty case, the cap creates a finite debt threshold even under the good equilibrium, $b_0^G = (s_{\max} - e)/r$. When the cap binds at the thresholds of both equilibria, that is, when $b_1^B = (s_{\max} + e)/R$ exceeds $b_{\max}$, debt dynamics above $b_{\max}$ are those of Section 3 with $s$ replaced by $s_{\max}$, and the three regimes of that section apply: the equilibrium is discontinuous under low uncertainty, continuous up to a jump under medium uncertainty, and unique when $e \geqslant \bar{e}(s_{\max})$, where $\bar{e}(s_{\max})$ is the threshold of Section 3 evaluated at $s_{\max}$. Since $\bar{e}$ is proportional to the surplus, uniqueness requires the cap to be low enough relative to the uncertainty, together with $e < s_{\max}$ for the surplus to remain positive. As $s_{\max} \to \bar{s}$ the equilibria converge to the baseline. As $s_{\max} \to \infty$ they converge to the uncapped Bohn rule, the cap no longer binds at $b_1^B$, and the unique-equilibrium regime disappears, consistent with Section 4.

# 6 Conclusions

We started this project with the prior that the range of multiplicity was typically very wide, and thus the risk of a shift to the bad equilibrium remained even at low levels of debt.

The analysis shows the answer to be different. The range depends very much on the degree of uncertainty about the course of primary balances and the strength of the feedback rule. With low uncertainty, the range is indeed wide, but it decreases and eventually disappears as uncertainty increases. This holds for uncertainty about the primary balance, provided the haircut in default is small enough relative to the safe rate, and for uncertainty about the strength of a temporary adjustment program, provided investors are optimistic

once the program is over. It does not hold for uncertainty about the primary balance under a permanent feedback rule, where the bad equilibrium survives at every degree of uncertainty, nor for increases in volatility that leave the range of outcomes unchanged, whose effect depends on how they alter the density of fiscal outcomes near the boundary between repayment and default. The underlying reason for what may be a counterintuitive result is that higher uncertainty decreases the strength of the interaction between the perceived and the actual probability of default. Part of the narrowing, however, comes from a lower level of debt being sustainable under optimism. Uncertainty does not make debt safer; it converts some self-fulfilling risk into fundamental risk at a lower debt limit.

Our model is too simple to have concrete policy implications. It has no intervention technology, no resource constraint for the central bank, and no welfare criterion. What it does have is a characterization of when an increase in the interest rate on government debt is informative about a shift in beliefs, and this is what the following remarks are anchored to.

Central banks have been given the responsibility to intervene when an increase in the interest rate on government bonds reflects liquidity rather than fundamental issues. In the terminology of our paper, they have been given the responsibility to enforce the good equilibrium and prevent the emergence of the bad equilibrium. Our model shows how difficult the diagnosis may be. With low uncertainty, an increase in the interest rate can only reflect a shift to the bad equilibrium. With medium uncertainty, it may reflect a deterioration of the good equilibrium, a shift to the bad one, or both, and only the first part is warranted by fundamentals. With high uncertainty, there is a single equilibrium and the increase reflects fundamentals alone. The diagnosis thus requires knowing which regime the economy is in, which in turn requires knowing the range of possible fiscal outcomes. This range is not observable, and the three regimes can produce the same observed increase in the rate. Assessing what the relevant situation is, and thus whether to intervene, is likely to be difficult at best.

A fiscal feedback rule changes the diagnosis in two ways. A rule strong enough to stabilize debt at the safe rate ensures that debt never explodes under the good equilibrium, so that, as long as the rule is unconstrained and the safe rate does not move, an increase in the rate on government debt can only reflect a shift to the bad equilibrium, and the stronger the rule, the higher the level of debt at which that shift can occur. Two qualifications matter, however. First, such a rule does not eliminate the bad equilibrium at any degree of uncertainty about the primary balance; only a rule strong enough to stabilize debt at the

pessimistic rate does. Second, once the fiscal effort is capped, as it must be at high enough debt, the good equilibrium again has a finite threshold, and at high levels of debt an increase in the rate may again reflect fundamentals rather than beliefs. The clean diagnosis holds only over the range of debt where the rule is credible and unconstrained.

Uncertainty about the strength of the rule itself reintroduces the three regimes of Section 3, with the degree of uncertainty about the feedback coefficient playing the role of uncertainty about the primary balance. Under our assumption that investors are optimistic once the program is over, enough uncertainty about the program eliminates multiplicity. If they are pessimistic in the continuation, the multiplicity of the continuation remains whatever the program. Thus, again, the degree of uncertainty matters very much, and so does what investors expect once the program ends.

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# A Nonuniform distribution of fiscal surpluses

This appendix derives the results behind the discussion of nonuniform surpluses in Section 5. Appendices A.1 to A.3 apply to any distribution of the primary surplus with a continuous and strictly positive density on $[s-e, s+e]$. Appendix A.4 specializes them to the truncated normal, and Appendix A.5 reports numerical results for our calibration. Throughout, $S$ denotes the primary surplus, $H$ its cumulative distribution function, $f$ its density and $q = H^{-1}$ its quantile function, so that $q(0) = s-e$ and $q(1) = s+e$. As in the main text, $1 + R(P) = (1+r)/(1-\theta P)$ and $R \equiv R(1) = (r+\theta)/(1-\theta)$.

## A.1 Fixed points for a given threshold

For a threshold $c$, the probability that debt next period exceeds $c$ is

$$F(b, c, P) = Prob\left(\left(1 + R(P)\right) b - S \geqslant c\right) = H\left(\left(1 + R(P)\right) b - c\right).$$

Define the inverse curve

$$b_c(P) \equiv \frac{c + q(P)}{1 + R(P)} = \frac{\left(c + q(P)\right)\left(1 - \theta P\right)}{1 + r}, \qquad P \in [0, 1]. \tag{14}$$

**Lemma 1.** *Fix $c$ and $b$.*

*(i) For $P \in (0,1)$, $F(b,c,P) \geqslant P$ if and only if $b \geqslant b_c(P)$, and $P$ is a fixed point of $F(b,c,\cdot)$ if and only if $b = b_c(P)$. $P = 0$ is a fixed point if and only if $b \leqslant b_c(0)$, and $P = 1$ is a fixed point if and only if $b \geqslant b_c(1)$.*

*(ii) The smallest and largest fixed points of $F(b,c,\cdot)$ are*

$$P_c^G(b) = \inf\left\{P \in [0,1) : b_c(P) \geqslant b\right\}, \qquad P_c^B(b) = \sup\left\{P \in (0,1] : b_c(P) \leqslant b\right\},$$

*with the conventions $\inf \varnothing = 1$ and $\sup \varnothing = 0$.*

*(iii) For $p \in (0,1]$, the lowest levels of debt at which these two schedules reach a probability $p$ are*

$$b_c^G(p) = \max_{P \in [0,p]} b_c(P), \qquad b_c^B(p) = \min_{P \in [p,1]} b_c(P).$$

*For $p = 0$, the same expressions give the highest levels of debt at which the two schedules*

*are still equal to zero,* $b_c^G(0) = b_c(0)$ *and* $b_c^B(0) = \min_{[0,1]} b_c$*. This is the convention used for* $b_0^G$ *and* $b_0^B$ *below; since it concerns a single point, the distance* (9) *is unaffected.*

*Proof.* (i) Since $H$ is strictly increasing on the support, for $P \in (0,1)$ we have $H(x) \geqslant P$ if and only if $x \geqslant q(P)$, with equality in one if and only if equality in the other. With $x = (1+R(P))b-c$, this is $b \geqslant b_c(P)$. $P = 0$ is a fixed point if and only if $(1+r)b-c \leqslant s-e$, and $P = 1$ is a fixed point if and only if $(1+R)b-c \geqslant s+e$, which are the two remaining statements. (ii) By (i), $F(b,c,P) > P$ for every $P$ below $P_c^G(b)$, and, by continuity of $b_c$, $P_c^G(b)$ is itself a fixed point; hence it is the smallest one. The argument for $P_c^B(b)$ is symmetric. (iii) Let $p > 0$. By (ii), $P_c^G(b) \geqslant p$ if and only if $b_c(P) < b$ for all $P < p$, and $P_c^B(b) \geqslant p$ if and only if $b_c(P) \leqslant b$ for some $P \geqslant p$. Taking the lowest such $b$ and using continuity gives the two expressions. For $p = 0$, by (i), $P_c^G(b) = 0$ if and only if $b \leqslant b_c(0)$ and $P_c^B(b) = 0$ if and only if $b < \min_{[0,1]} b_c$, which gives the stated convention. ■

Since investors locate the fixed point by iterating from $P^e = 0$ or $P^e = 1$, and $F$ is nondecreasing in $P^e$, the good equilibrium selects the smallest fixed point and the bad equilibrium the largest one. Lemma 1(iii) gives the functions $b^G(p)$ and $b^B(p)$ used in the distance (9), with $c = b_1^G$ and $c = b_1^B$ respectively.

## A.2 Thresholds

As in the main text, the threshold of each equilibrium is the lowest level of debt above which the corresponding probability of default is equal to one.

**Proposition 7.** *(i) For any distribution of the surplus, the threshold of the bad equilibrium is* $b_1^B = (s+e)/R$*.*

*(ii) The threshold of the good equilibrium is*

$$b_1^G = \max_{P \in [0,1]} \frac{q(P)}{R(P)}, \tag{15}$$

*and it is the only threshold consistent with the selection of the smallest fixed point.*

*(iii)* $b_1^G \geqslant b_1^B$*,* $b_0^G = b_{b_1^G}(0) = (b_1^G + s - e)/(1+r)$*, and* $b_0^B = \min_{P \in [0,1]} b_{b_1^B}(P)$*.*

*Proof.* (i) By Lemma 1, $P_c^B(b) = 1$ if and only if $b \geqslant b_c(1) = (c+s+e)/(1+R)$. Consistency requires $c = (c+s+e)/(1+R)$, that is, $c = (s+e)/R$. (ii) By Lemma 1, $P_c^G(b) = 1$ for

all $b > M(c) \equiv \max_{P\in[0,1]} b_c(P)$, while $P_c^G(b) < 1$ for $b < M(c)$. Consistency thus requires $c = M(c)$ or, equivalently,

$$\max_{P\in[0,1]} \frac{q(P) - cR(P)}{1 + R(P)} = 0.$$

The left-hand side is continuous and strictly decreasing in $c$, because $R(P) \geqslant r > 0$, so there is a unique solution, which is (15). (iii) The first statement follows from evaluating (15) at $P = 1$, and the others from the $p = 0$ convention in Lemma 1(iii). ■

Under the uniform distribution, $q(P) = s - e + 2eP$, and the maximum in (15) is attained at $P = 0$ when $e \leqslant e_G$, at an interior tangency when $e_G < e < \bar{e}$, and at $P = 1$ when $e \geqslant \bar{e}$, which gives the thresholds of Section 3.

## A.3 A necessary and sufficient condition for uniqueness

Define

$$L_c(x) \equiv \frac{\theta\,(c + x)\,f(x)}{1 - \theta H(x)}, \qquad x \in [s - e, s + e]. \tag{16}$$

**Proposition 8.** *The good and bad equilibria coincide if and only if $L_{b_1^B}(x) \leqslant 1$ for all $x \in [s - e, s + e]$.*

*Proof.* Let $c = b_1^B$. Differentiating (14), and using $q'(P) = 1/f(q(P))$ and $H(q(P)) = P$,

$$b_c'(P) = \frac{1 - \theta H(x)}{(1 + r)f(x)}\left[1 - L_c(x)\right], \qquad x = q(P).$$

Suppose $L_c \leqslant 1$ on the support. Then $b_c$ is nondecreasing, so $b_c(P) \leqslant b_c(1) = c$ for all $P$, which is equivalent to $q(P) \leqslant cR(P)$; by (15), $b_1^G = c = b_1^B$. The two equilibria then share the same inverse curve and, since it is nondecreasing, $b_c^G(p) = b_c^B(p) = b_c(p)$ for all $p$: the two schedules coincide. Conversely, suppose $L_c(x) > 1$ for some $x$. By continuity, $b_c$ is strictly decreasing on some interval $(P_1, P_2)$. Either $\max_P b_c(P) > c$, in which case $b_1^G > b_1^B$, or $b_1^G = b_1^B = c$, in which case, for $p \in (P_1, P_2)$, $b_c^G(p) \geqslant b_c(P_1) > b_c(p) \geqslant b_c^B(p)$. In both cases, the two schedules differ over an interval of debt. ■

Two remarks follow. First, at the top of the support $H = 1$; using $b_1^B + s + e = (1 + R)\,b_1^B$ and $1 + R = (1 + r)/(1 - \theta)$ gives $L_{b_1^B}(s + e) = F'\left(b_1^B, b_1^B, 1\right)$. The condition $F'\left(b_1^B, b_1^B, 1\right) \leqslant 1$ used in the main text is therefore necessary for uniqueness. Second, under the uniform distribution, $f = 1/(2e)$ and $H(x) = (x - s + e)/(2e)$, so $L_c$ is increasing in $x$ and reaches

its maximum at $x = s + e$. The condition is then also sufficient, and it is equivalent to $e \geqslant \bar{e}$. Under other distributions, $L_c$ need not be maximized at the top of the support, and uniqueness can fail even when $F'\left(b_1^B, b_1^B, 1\right) < 1$.

## A.4 The truncated normal

Let $S$ follow a normal distribution with location $s$ and scale $\sigma$, truncated to $[s-e, s+e]$, and let $t \equiv e/\sigma$. Its density is

$$f(x) = \frac{\varphi\left(\frac{x-s}{\sigma}\right)}{\sigma\left[\Phi(t) - \Phi(-t)\right]}, \qquad x \in [s-e, s+e].$$

**Lemma 2.** *$F'\left(b_1^B, b_1^B, 1\right) = \Delta(t)\, F'_{uniform}\left(b_1^B, b_1^B, 1\right)$, with $\Delta(t) = 2t\varphi(t)/\left[\Phi(t) - \Phi(-t)\right]$. The function $\Delta$ is strictly decreasing, satisfies $\Delta(t) < 1$ for all $t > 0$, $\lim_{t\to 0} \Delta(t) = 1$ and $\lim_{t\to\infty} \Delta(t) = 0$.*

*Proof.* $F'\left(b_1^B, b_1^B, 1\right)$ is proportional to the density of the surplus at $s + e$, and $f(s+e) = \Delta(t)/(2e)$, where $1/(2e)$ is the uniform density. Let $u(t) = 2\Phi(t) - 1 - 2t\varphi(t)$. Then $u(0) = 0$ and $u'(t) = 2t^2\varphi(t) > 0$, so $u(t) > 0$ for $t > 0$, which is $\Delta(t) < 1$. Next, $\Delta'(t)$ has the sign of $v(t) = (1-t^2)\left[2\Phi(t) - 1\right] - 2t\varphi(t)$. Since $v(0) = 0$ and $v'(t) = -2t\left[2\Phi(t) - 1\right] < 0$ for $t > 0$, $v(t) < 0$ for $t > 0$. The limits follow from $2\Phi(t) - 1 \sim 2t\varphi(0)$ as $t \to 0$ and $t\varphi(t) \to 0$ as $t \to \infty$. ■

**Lemma 3.** *$\mathrm{Var}(S)$ is strictly increasing in $\sigma$, and it converges to $e^2/3$, the variance of the uniform distribution on $[s-e, s+e]$, as $\sigma \to \infty$.*

*Proof.* Let $Y = S - s$, whose density is proportional to $\exp\left(-y^2/(2\sigma^2)\right)$ on $[-e, e]$. By symmetry, $\mathbb{E}[Y] = 0$ and $\mathrm{Var}(S) = \mathbb{E}[Y^2]$. Differentiating under the integral sign, and since the derivative of the log density with respect to $\sigma$ is $y^2/\sigma^3$ up to a constant,

$$\frac{d\,\mathbb{E}[Y^2]}{d\sigma} = \mathrm{Cov}\left(Y^2, \frac{Y^2}{\sigma^3}\right) = \frac{\mathrm{Var}(Y^2)}{\sigma^3} > 0.$$

As $\sigma \to \infty$, the density converges uniformly to $1/(2e)$. ■

Lemma 3 ensures that $\sigma$ is a genuine measure of dispersion: for a given support, a higher $\sigma$ raises the variance of the surplus, and the uniform distribution is the limit of maximal dispersion.

**Proposition 9.** *(i) If $e > \bar{e}$, the equilibrium is unique for all $\sigma$ large enough. If $e < \bar{e}$, there are multiple equilibria for all $\sigma$ large enough.*

*(ii) As $\sigma \to 0$, $b_1^G \to \max\{s/r, (s+e)/R\}$. Hence, if $rb_1^B < s$, or equivalently $e < (R/r - 1)\, s$, there are multiple equilibria for all $\sigma$ small enough, and $b_1^G - b_1^B \to s/r - (s+e)/R > 0$.*

*Proof.* (i) As $\sigma \to \infty$, $f$ and $H$ converge uniformly on $[s-e, s+e]$ to the density and the cumulative distribution function of the uniform distribution, so $L_{b_1^B}$ converges uniformly to its uniform counterpart. By the remarks after Proposition 8, the maximum of the latter is $F'_{uniform}\left(b_1^B, b_1^B, 1\right)$, which is strictly smaller than one if $e > \bar{e}$ and strictly larger than one if $e < \bar{e}$. The result follows from Proposition 8. (ii) For fixed $P \in (0,1)$, $q(P) \to s$ as $\sigma \to 0$. Hence, by (15), $\liminf_{\sigma\to 0} b_1^G \geqslant s/R(P)$ for every $P \in (0,1)$ and, letting $P \to 0$, $\liminf_{\sigma\to 0} b_1^G \geqslant s/r$; also, $b_1^G \geqslant b_1^B = (s+e)/R$ for every $\sigma$. For the upper bound, fix $\delta \in (0,1)$. For $P \leqslant 1-\delta$, $q(P)/R(P) \leqslant q(1-\delta)/r$, which converges to $s/r$; for $P > 1-\delta$, $q(P)/R(P) \leqslant (s+e)/R(1-\delta)$. Hence $\limsup_{\sigma\to 0} b_1^G \leqslant \max\{s/r, (s+e)/R(1-\delta)\}$ for every $\delta$, and letting $\delta \to 0$ gives the result. ■

The limit in (ii) differs from the case $\sigma = 0$, in which the surplus equals $s$ with probability one and $b_1^B = s/R$: for every $\sigma > 0$, the support of the surplus remains $[s-e, s+e]$, and the bad threshold remains $(s+e)/R$. This has a consequence for the interpretation of the small-$\sigma$ results in Appendix A.5. Since certain default under pessimism requires the best possible surplus, $s+e$, to be insufficient, the bad threshold is pinned by the edge of the support even when the probability mass near that edge is negligible. The limit of the threshold gap as $\sigma \to 0$, $s/r - (s+e)/R$, is therefore smaller than its value under certainty, $s/r - s/R$, for a reason that has nothing to do with dispersion. The distance $W$ does not undo this. In the same limit, the bad schedule rises to values close to one at $\left(b_1^B + s\right)(1-\theta)/(1+r)$, below $b_1^B$, so $W$ exceeds the threshold gap, but only by a small amount: at $e = 0.06$ and $\sigma = 0.012$, $W = 0.53$ against a gap of 0.50. Comparisons between small $\sigma$ and the certainty case thus mix the effect of the support with that of dispersion, and the small-$\sigma$ results should be read as the case of a tiny dispersion with the support held fixed.

## A.5 Numerical results

We now use Propositions 7 and 8 to classify the equilibria for our calibration, $\theta = 0.1$, $r = 0.05$ and $s = 0.1$, for which $e_B = 0.0538$ and $\bar{e} = 0.0636$. For each pair $(e, \sigma)$ there

are three possible outcomes. The thresholds of the two equilibria may differ, $b_1^G > b_1^B$. The thresholds may coincide while $\max_x L_{b_1^B}(x) > 1$, so that the two schedules differ below the common threshold. Or the equilibrium may be unique. Figure 14 shows the outcome over a grid of values of $e$ and $t = e/\sigma$, and Table A.1 reports it for selected values. Because the classification is categorical, Figure 15 complements it with the distance $W$ of (9) over the same grid, which weights each level of debt by the difference between the two equilibria and thus measures their economic size rather than their mere existence.

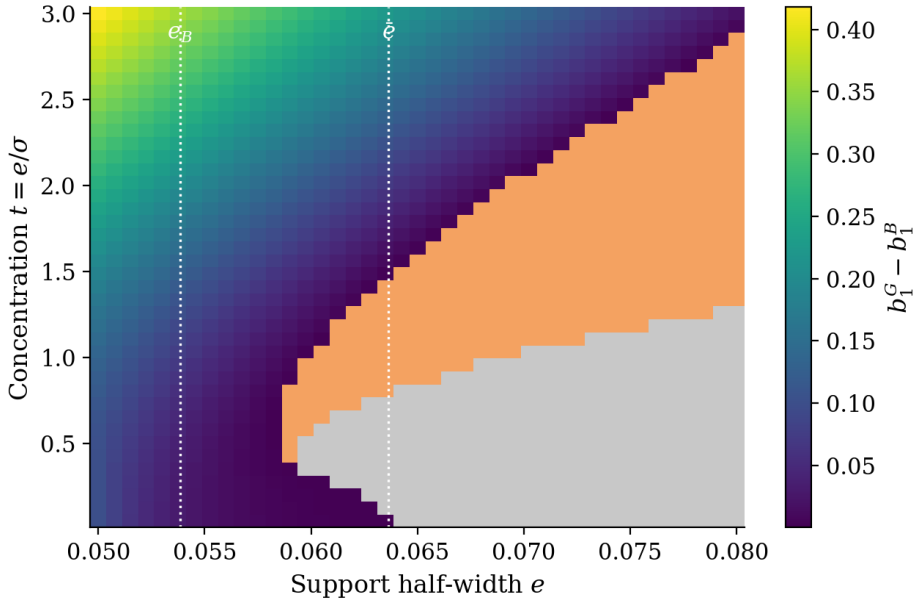



**Figure 14:** Multiplicity under a truncated normal surplus. Colored region: $b_1^G > b_1^B$, with the color giving $b_1^G - b_1^B$. Orange: $b_1^G = b_1^B$, but the two schedules differ below the common threshold. Grey: unique equilibrium. The bottom of the figure, $t \to 0$, corresponds to the uniform distribution. Parameters: $\theta = 0.1$, $r = 0.05$, $s = 0.1$.

The second outcome is illustrated in Figure 16 for $e = 0.06$ and $\sigma = 0.10$. The two thresholds coincide at $b_1^B = b_1^G = 0.96$, but the inverse curve $b_c(P)$ is not monotone: at $b = 0.95909$, $F(b, b_1^B, \cdot)$ has three fixed points, 0.541, 0.664 and 0.781, so the good and bad schedules assign different probabilities of default to the same level of debt. For $\sigma = 0.15$, instead, the inverse curve is increasing and the equilibrium is unique: $\max_x L_{b_1^B}(x) \approx 0.991$, and the bound $\max_x L_{b_1^B}(x) < 0.993$ can be established with exact rational arithmetic, by bounding $L_{b_1^B}$ from above on each cell of a fine partition of the support.

Four results stand out. First, for values of $e$ below about 0.058, there are multiple equilibria for every value of $\sigma$. Second, for values of $e$ between about 0.058 and $\bar{e}$, there

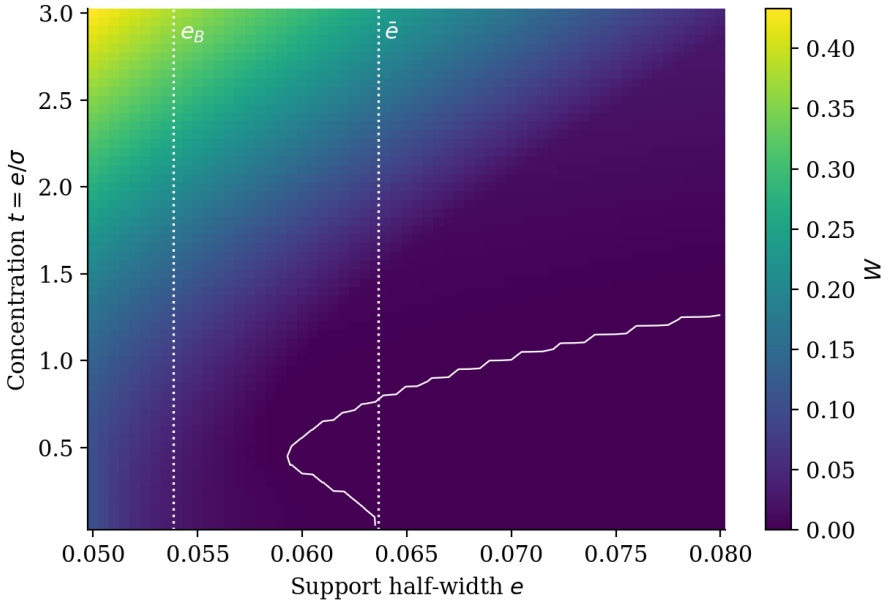



**Figure 15:** Distance $W$ between the good and bad schedules under a truncated normal surplus, over the same grid as Figure 14. The white contour encloses the region where $W = 0$ and the equilibrium is unique. Dotted vertical lines mark $e_B$ and $\bar{e}$. Parameters: $\theta = 0.1$, $r = 0.05$, $s = 0.1$.

**Table A.1:** Equilibria under a truncated normal surplus

| | | $e = 0.060$ | | Outcome | | | |
|---|---|---|---|---|---|---|---|
| $t = e/\sigma$ | $\sigma$ at $e = 0.060$ | $b_1^G - b_1^B$ | $\max_x L_{b_1^B}(x)$ | $e = 0.056$ | $e = 0.060$ | $e = 0.063$ | $e = 0.066$ |
| 0.05 | 1.200 | 0.0024 | 1.036 | M | M | M | U |
| 0.10 | 0.600 | 0.0023 | 1.034 | M | M | M | U |
| 0.20 | 0.300 | 0.0015 | 1.023 | M | M | U | U |
| 0.30 | 0.200 | 0.0003 | 1.006 | M | M | U | U |
| 0.40 | 0.150 | 0 | 0.991 | M | U | U | U |
| 0.50 | 0.120 | 0 | 0.993 | M | U | U | U |
| 0.75 | 0.080 | 0 | 1.030 | M | F | U | U |
| 1.00 | 0.060 | 0 | 1.094 | M | F | F | F |
| 1.25 | 0.048 | 0.0178 | 1.182 | M | M | F | F |
| 1.50 | 0.040 | 0.0447 | 1.289 | M | M | M | F |
| 2.00 | 0.030 | 0.1143 | 1.559 | M | M | M | M |
| 3.00 | 0.020 | 0.2723 | 2.236 | M | M | M | M |
| 5.00 | 0.012 | 0.4976 | 3.715 | M | M | M | M |
| Uniform | $\infty$ | 0.0025 | 1.037 | M | M | M | U |

M: the thresholds of the two equilibria differ. F: the thresholds coincide, but the two schedules differ below the common threshold. U: unique equilibrium. Parameters: $\theta = 0.1$, $r = 0.05$, $s = 0.1$.

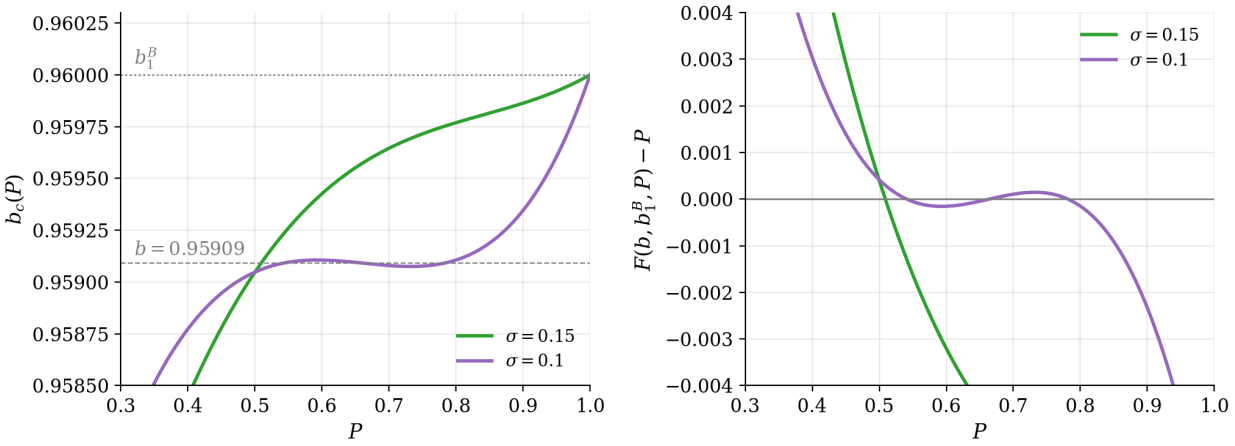



**Figure 16:** Same threshold, different schedules ($e = 0.06$). Left: inverse curves $b_c(P)$, with $c = b_1^B = 0.96$, for $\sigma = 0.15$ (unique equilibrium) and $\sigma = 0.10$. Right: $F(b, b_1^B, P) - P$ at $b = 0.95909$; its zeros are the fixed points. Parameters: $\theta = 0.1$, $r = 0.05$, $s = 0.1$.

are multiple equilibria under the uniform distribution, but the equilibrium is unique for intermediate values of $\sigma$; at $e = 0.06$, this is the case for $\sigma \in (0.104, 0.182)$. As $\sigma$ decreases from high values, the lower density at the top of the support first eliminates multiplicity, as implied by Lemma 2; as $\sigma$ decreases further, the concentration of mass around $s$ brings it back, first through a fold of the inverse curve and then through a gap between the thresholds. Third, for $e > \bar{e}$, the equilibrium is unique if and only if $\sigma$ is large enough. Fourth, the uniform distribution is not a lower bound for multiplicity: at $e = 0.056$, the gap $b_1^G - b_1^B$ reaches 0.0140 at $\sigma \approx 0.2$, below its value of 0.0144 under the uniform distribution. Fifth, Figure 15 shows that the F region, and the part of the M region close to the uniqueness boundary, involve differences between the two schedules that are economically small: $W$ is close to zero there, and it becomes large only for high concentration and small $e$, where the two thresholds are far apart. The categorical map thus overstates the size of multiplicity near its boundary, in the same way that the width of the support does in Section 3.



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